Journal of Approximation Theory **94**, 333–382 (1998)
Article No. AT983199

Jackson Theorems for Erdős Weights in $L_p(0<p\leqslant\infty)$ [1]

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Communicated by Dany Leviatan

Received May 22, 1995; accepted in revised form June 4, 1997

An *Erdős weight* is of the form $W:=e^{-Q}$ where Q is even and of faster than polynomial growth at ∞. For example, we can take

$$Q(x):=\exp_k(|x|^\alpha), \qquad k\geqslant 1, \quad \alpha>0, \quad x\in\mathbb{R},$$

where $\exp_k$ denotes the kth iterated exponential. We prove Jackson theorems in weighted L_p spaces with norm $\|fW\|_{L_p(\mathbb{R})}$ for all $0<p\leqslant\infty$. These are the first proper Jackson theorems for Erdős weights even in L_∞. An interesting feature is a Timan–Nikolskii–Brudnyi effect: The degree of approximation improves towards the endpoints of a certain interval. By contrast, there is no such feature for Freud weights. © 1998 Academic Press

Key Words: Erdős weights; Jackson theorem; polynomial approximation.

1. STATEMENT OF RESULTS

In recent years, there have been many advances in the theory of weighted polynomial approximation, and orthonormal polynomials, associated with the weights

$$W:=e^{-Q}. \tag{1.1}$$

Here $Q\colon\mathbb{R}\to\mathbb{R}$ is even, and typically grows at least as fast as $|x|^\lambda$, some $\lambda>1$, at infinity. In some contexts, there has been a distinction between the case where Q is of polynomial growth at infinity (the so-called *Freud case*) and where Q is of faster than polynomial growth at infinity (the so-called *Erdős case*). To some extent, this is similar to the distinction between entire functions of finite, and infinite, order. For further orientation on this topic, see [7, 10, 15, 16, 21, 22].

[1] This article is a companion to "Converse and Smoothness Theorems for Erdős Weights in L_p $(0<p\leqslant\infty)$" by S. B. Damelin which ran in Journal of Approximation Theory **93**, 349–398.

0021-9045/98 $25.00

In this paper, we discuss Jackson theorems for Erdős weights. That is, we estimate

$$E_n[f]_{W,p} := \inf_{P \in \mathcal{P}_n} \|(f-P)W\|_{L_p(\mathbb{R})}, \tag{1.2}$$

$0 < p \leqslant \infty$, where the $\mathcal{P}_n$ denote the polynomials of degree at most n.

Our methods are similar to those in [6], where Jackson theorems were proved for Freud weights. The approach involves approximating f by a spline (or piecewise polynomial), representing the piecewise polynomial in terms of certain characteristic functions, and then approximating the characteristic functions (in a suitable sense) by polynomials. This method has the advantage of involving only hypotheses on Q', in contrast with the more complicated approach via orthogonal polynomials and de la Vallee Poussin sums, that typically involves hypotheses on Q'' [7, 10, 17, 21]. In the Erdős weight context, some new features arise: the degree of approximation improves toward the endpoints of the Mhaskar–Saff interval, and to reflect this, we need a more complicated modulus of continuity, and some proofs become more involved.

To state our result, we need to define our class of weights, as well as various quantities. First, we say that a function $f\colon (a, b) \to (0, \infty)$ is *quasi-increasing* if $\exists C > 0$ such that

$$a < x < y < b \Rightarrow f(x) \leqslant Cf(y).$$

DEFINITION 1.1. Let $W := e^{-Q}$, where

(a) $Q\colon \mathbb{R} \to \mathbb{R}$ is even, continuous, and Q' is positive in $(0, \infty)$.

(b) $xQ'(x)$ is strictly increasing in $(0, \infty)$ with right limit 0 at 0.

(c) The function

$$T(x) := \frac{xQ'(x)}{Q(x)} \tag{1.3}$$

is quasi-increasing in (C, ∞) for some $C > 0$, and

$$\lim_{x \to \infty} T(x) = \infty. \tag{1.4}$$

(d) There exist $C_1, C_2, C_3 > 0$ such that

$$\frac{yQ'(y)}{xQ'(x)} \leqslant C_1 \left(\frac{Q(y)}{Q(x)}\right)^{C_2}, \qquad y \geqslant x \geqslant C_3. \tag{1.5}$$

Then we write $W = e^{-Q} \in \mathcal{E}_1$.

The archetypal example of $W \in \mathscr{E}_1$ is

$$W(x) := W_{k,\alpha}(x) := \exp(-\exp_k(|x|^\alpha)), \qquad k \geqslant 1, \quad \alpha > 0, \tag{1.6}$$

where $\exp_k = \exp(\exp(\cdots))$ denotes the kth iterated exponential. For this weight, we see

$$T(x) = \alpha x^\alpha \prod_{j=1}^{k-1} \exp_j(x^\alpha), \qquad x > 0.$$

It is not too difficult to see that we can choose $C_2 > 1$ in (1.5) arbitrarily close to 1 in this case. Another example is

$$W(x) := \exp(-\exp[\log(2+x^2)]^\beta), \qquad \beta > 1.$$

Here

$$T(x) = \frac{2\beta x^2}{2+x^2}[\log(2+x^2)]^{\beta-1}, \qquad x > 0.$$

Again, we can choose C_2 arbitrarily close to 1.

The function T measures the regularity of growth of Q. In particular, (1.4) forces Q to be of faster than polynomial growth at ∞. The reader is cautioned that in other papers on Erdős weights [14, 17] the function

$$T_1(x) := 1 + \frac{xQ''(x)}{Q'(x)}$$

was used (and denoted by T), but it has essentially the same rate of growth as T, for "nice" weights.

We need the condition that $xQ'(x)$ be strictly increasing to guarantee the existence of the *Mhaskar–Rakhmanov–Saff number* a_u, the positive root of the equation

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)\frac{dt}{\sqrt{1-t^2}}, \qquad u > 0. \tag{1.7}$$

If we used something other than a_u, we could require less of $xQ'(x)$, namely that it be quasi-increasing for large x. However, this would complicate formulations, so is omitted. For those to whom a_u is new, its significance lies partly in the identity [18–20]

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n, a_n]}, \qquad P \in \mathscr{P}_n, \tag{1.8}$$

and that a_n is the "smallest" such number.

Our modulus of continuity involves two parts, a "main part" and a "tail." The "main part" involves rth symmetric differences over a suitable interval, and the tail involves an error of weighted polynomial approximation over the remainder of the real line. The size of this "suitable interval" is determined by the decreasing function of t,

$$\sigma(t) := \inf\left\{a_u : \frac{a_u}{u} \leqslant t\right\}, \qquad t > 0. \tag{1.9}$$

Thus σ is essentially the inverse function of the function $u \to a_u/u$, which decays to 0 as $u \to \infty$.

For $h > 0$, an interval J, and $r \geqslant 1$, we define the rth symmetric difference

$$\Delta_h^r(f, x, J) := \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x + \frac{rh}{2} - ih\right), \tag{1.10}$$

provided all arguments of f lie in J, and 0 otherwise. Sometimes the increment h will depend on x, and on the function

$$\Phi_t(x) := \sqrt{\left|1 - \frac{|x|}{\sigma(t)}\right|} + T(\sigma(t))^{-1/2}, \qquad x \in \mathbb{R}. \tag{1.11}$$

This is the case in our modulus of continuity

$$\omega_{r,p}(f, W, t) := \sup_{0 < h \leqslant t} \| W \Delta^r_{h\Phi_t(x)}(f, x, \mathbb{R}) \|_{L_p(|x| \leqslant \sigma(2t))} + \inf_{P \in \mathcal{P}_{r-1}} \|(f - P) W\|_{L_p(|x| \geqslant \sigma(4t))} \tag{1.12a}$$

and its averaged "cousin"

$$\bar{\omega}_{r,p}(f, W, t) := \left(\frac{1}{t}\int_0^t \| W \Delta^r_{h\Phi_t(x)}(f, x, \mathbb{R}) \|^p_{L_p(|x| \leqslant \sigma(2t))}\, dh\right)^{1/p} + \inf_{P \in \mathcal{P}_{r-1}} \|(f - P) W\|_{L_p(|x| \geqslant \sigma(4t))}. \tag{1.12b}$$

If $p = \infty$, we set

$$\bar{\omega}_{r,p}(f, W, ;\,) = \omega_{r,p}(f, W, ;\,).$$

Observe that

$$\bar{\omega}_{r,p}(f, W, t) \leqslant \omega_{r,p}(f, W, t)$$

for every fixed $t \in \mathbb{R}$.

The inf in the tail is at first disconcerting, but note that it is over polynomials of degree at most $r-1$, not n. Its presence ensures that for $f \in \mathscr{P}_{r-1}$, $\omega_{r,p}(f, W, t) \equiv 0$. The modulus of continuity is rather difficult to assimilate (as is the case with all its cousins [6, 7] for weighted approximation on $\mathbb{R}$). A good way to view the function σ is that for purposes of approximation by polynomials of degree at most n, essentially $t = a_n/n$, the main part of the modulus is taken over the range $[-a_{n/2}, a_{n/2}]$, and the tail is taken over $\mathbb{R} \backslash [-a_{n/2}, a_{n/2}]$. Moreover, the function Φ_t describes the improvement in the degree of approximation near $\pm a_{n/2}$, in much the same way that $\sqrt{1-x^2}$ does for weights on $[-1, 1]$.

It is possible to replace $\sigma(2t)$ by the somewhat larger term $\sigma(t) - At$ and $\sigma(4t)$ by the somewhat smaller term $\sigma(t) - Bt$ for suitable A, B in our modulus, under additional conditions on Q. However, it hardly seems worth the effort, as the resulting modulus is almost certainly equivalent to the above one. As evidence of this, we note that in [3], the first author proves that the above modulus is equivalent to a natural K-functional/realization functional.

The following is our main Jackson theorem:

THEOREM 1.2. *Let* $W := e^{-Q} \in \mathscr{E}_1$. *Let* $r \geqslant 1$ *and* $0 < p \leqslant \infty$. *Then for* $f\colon \mathbb{R} \to \mathbb{R}$ *for which* $fW \in L_p(\mathbb{R})$ (*and for* $p = \infty$, *we require* f *to be continuous, and* fW *to vanish at* $\pm\infty$), *we have for* $n \geqslant C_3$,

$$E_n[f]_{W,p} \leqslant C_1 \bar{\omega}_{r,p}\left(f, W, C_2 \frac{a_n}{n}\right) \leqslant C_1 \omega_{r,p}\left(f, W, C_2 \frac{a_n}{n}\right), \tag{1.13}$$

where the C_j, $j = 1, 2, 3$, *do not depend on* f *or* n.

Remark. We remark that it is possible using the methods of [3, 6] to prove Theorem 1.2 for $n \geqslant r-1$.

Unfortunately, the modulus $\omega_{r,p}(f, W, t)$ is not obviously monotone increasing in t. So we also present a result involving the increasing modulus

$$\omega^*_{r,p}(f, W, t) := \sup_{\substack{0<h\leqslant t \\ 0<\tau\leqslant L}} \| W \Delta^r_{\tau h \Phi_h(x)}(f, x, \mathbb{R}) \|_{L_p(|x| \leqslant \sigma(2h))} + \inf_{P \in \mathscr{P}_{r-1}} \| (f-P) W \|_{L_p(|x| \geqslant \sigma(4t))}. \tag{1.14}$$

Here L is a fixed (large enough) number independent of f, t.

THEOREM 1.3. *Under the hypotheses of Theorem* 1.2,

$$E_n[f]_{W,p} \leqslant C_3 \omega^*_{r,p}\left(f, W, C_4 \frac{a_n}{n}\right), \tag{1.15}$$

where the C_j, $j=3,4$ *do not depend on* f *or* n.

It seems likely that one should only really need $\tau = L$ in the definition of $\omega^*_{r,p}$, but we have only been able to prove this under additional conditions, see Section 7.

The modulus of continuity is analyzed in [3], and in particular the relationship to K-functionals/realization functionals is discussed. These have the consequence, that at least for $p \geqslant 1$, we can dispense with the constant C_2 inside the modulus $\omega_{r,p}$ in (1.13) or (1.15). For $p<1$, this requires extra hypotheses on W.

The paper is organized as follows: In Section 2, we present some technical details related to Q, a_u, and so on. In Section 3, we present estimates involving $\sigma(t)$ and Φ_t. In Section 4, we obtain polynomial approximations to W^{-1} over suitable intervals, and then in Section 5, we present our crucial approximations to characteristic functions. We prove Theorem 1.2 in Section 6 and Theorem 1.3 in Section 7. Moreover, we discuss simplification of the modulus $\omega^*_{r,p}$ in Section 7.

We close this section with a little more notation. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x and $P \in \mathscr{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C \neq C(L)$ to indicate that C is independent of L. Let (c_n), (d_n) be real sequences. The notation $c_n \sim d_n$ means that $C_1 \leqslant c_n/d_n \leqslant C_2$ for the relevant range of n. Similar notation is used for functions and sequences of functions. In the sequel, we assume that $W = e^{-Q} \in \mathscr{E}_1$.

2. TECHNICAL LEMMAS

LEMMA 2.1. (a) *For some* C_j, $j=1,2,3$, *and* $s \geqslant r \geqslant C_3$,

$$\left(\frac{s}{r}\right)^{C_2 T(r)} \leqslant \frac{Q(s)}{Q(r)} \leqslant \left(\frac{s}{r}\right)^{C_1 T(s)}. \tag{2.1}$$

Moreover,

$$\left(\frac{s}{r}\right)^{C_2 T(r)} \frac{T(s)}{T(r)} \leqslant \frac{sQ'(s)}{rQ'(r)} \leqslant \frac{T(s)}{T(r)} \left(\frac{s}{r}\right)^{C_1 T(s)}. \tag{2.2}$$

(b) *Given* $\delta > 0$, *there exists* C *such that*

$$T(y) \sim T\left(y\left(1 - \frac{\delta}{T(y)}\right)\right), \qquad y \geqslant C. \tag{2.3}$$

(c) *Given* $A \geqslant 0$, *the functions* $Q'(u)\,u^{-A}$ *and* $Q(u)\,u^{-A}$ *are quasi-increasing for large enough* u.

Proof. (a) Firstly, (2.1) follows from the identity

$$\log \frac{Q(s)}{Q(r)} = \int_r^s \frac{T(t)}{t}\,dt$$

and the fact that T is quasi-increasing. Then the definition (1.3) of T gives (2.2).

(b) We can reformulate (1.5) as

$$\frac{T(y)}{T(x)} \leqslant C_1 \left(\frac{Q(y)}{Q(x)}\right)^{C_2 - 1}.$$

Hence for $x = y(1 - \delta T/(y))$, the quasi-increasing nature of T gives

$$C_4 \leqslant \frac{T(y)}{T(x)} \leqslant C_1 \exp\left((C_2 - 1)\int_x^y \frac{T(t)}{t}\,dt\right)$$

$$\leqslant C_1 \exp\left(C_5 T(y) \log \frac{y}{x}\right) \leqslant C_6.$$

Recall here that $T(y)$ is large for large y.

(c) From (2.2) if $s \geqslant r \geqslant C$,

$$\frac{Q'(s)\,s^{-A}}{Q'(r)\,r^{-A}} \geqslant \frac{T(s)}{T(r)} \left(\frac{s}{r}\right)^{C_2 T(r) - 1 - A} \geqslant C_7.$$

Here we have used the quasi-monotonicity of T, and also that if C is large enough, then $C_2 T(r) - 1 - A \geqslant 0$ and similarly for $Q(s)s^{-A}$. ∎

Next, some properties of a_u:

LEMMA 2.2. (a) *a_u is uniquely defined and continuous for $u \in (0, \infty)$, and is a strictly increasing function of u.*

(b) *For* $u \geqslant C$,

$$a_u Q'(a_u) \sim u T(a_u)^{1/2}; \tag{2.4}$$

$$Q(a_u) \sim u T(a_u)^{-1/2}. \tag{2.5}$$

(c) *Given fixed* $\beta > 0$, *we have for large* u,

$$T(a_{\beta u}) \sim T(a_u). \tag{2.6}$$

(d) *Given fixed* $\alpha > 1$,

$$\frac{a_{\alpha u}}{a_u} - 1 \sim \frac{1}{T(a_u)}. \tag{2.7}$$

(e) *If* C_2 *is as in* (1.5),

$$T(a_u) \leqslant C_6 u^{2(C_2-1)/(C_2+1)} = C_6 u^{2(1-\delta)} \tag{2.8}$$

with $\delta > 0$.

(f) *If* $\alpha > 1$, *then for large enough* u,

$$\frac{Q(a_{\alpha u})}{Q(a_u)} \geqslant C_7 > 1. \tag{2.9}$$

(g) *For some* C_8, C_9, C_{10}, C_{11}, C_{12}, $u \geqslant C_8$, *and* $L \geqslant 1$,

$$\exp\left(C_{11}\frac{\log(C_{12}L)}{T(a_u)}\right) \geqslant \frac{a_{Lu}}{a_u} \geqslant 1 + C_9\frac{\log(C_{10}L)}{T(a_{Lu})}. \tag{2.10}$$

Proof. (a) The function $u \to a_u$ is the inverse of the strictly increasing continuous function

$$a \to \frac{2}{\pi}\int_0^1 atQ'(at)\frac{dt}{\sqrt{1-t^2}}\,dt, \qquad a \in (0, \infty),$$

which has right limit 0 at 0 and limit ∞ at ∞. (Note that this function is continuous even if Q' is not.) So the assertion follows.

(b) For u so large that $T(a_u) > 2$, we have

$$\begin{aligned}
\frac{u}{a_u Q'(a_u)} &= \frac{2}{\pi}\left[\int_0^{1-1/T(a_u)} + \int_{1-1/T(a_u)}^1\right]\frac{a_u t Q'(a_u t)}{a_u Q'(a_u)}\frac{dt}{\sqrt{1-t^2}} \\
&\leqslant \frac{2}{\pi}T(a_u)^{1/2}\int_0^{1-1/T(a_u)}\frac{a_u Q'(a_u t)}{a_u Q'(a_u)}\,dt + \frac{2}{\pi}\int_{1-1/T(a_u)}^1\frac{dt}{\sqrt{1-t^2}} \\
&\leqslant \frac{2}{\pi}T(a_u)^{1/2}\frac{Q(a_u)-Q(0)}{a_u Q'(a_u)} + \frac{4}{\pi}T(a_u)^{-1/2} \\
&\leqslant \frac{4}{\pi}T(a_u)^{1/2}\frac{Q(a_u)}{a_u Q'(a_u)} + \frac{4}{\pi}T(a_u)^{-1/2} = \frac{8}{\pi}T(a_u)^{-1/2}.
\end{aligned}$$

Here we also need u so large that $Q(a_u) \geqslant |Q(0)|$. So we have

$$a_u Q'(a_u) \geqslant \frac{\pi}{8}uT(a_u)^{1/2}.$$

In the other direction, (2.2) gives for large u,

$$\begin{aligned}
\frac{u}{a_u Q'(a_u)} &= \frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{a_u Q'(a_u)} \frac{dt}{\sqrt{1-t^2}} \\
&\geqslant C_1 \int_{1/2}^1 \frac{T(a_u t)}{T(a_u)} t^{C_1 T(a_u)} \frac{dt}{\sqrt{1-t^2}} \\
&\geqslant C_2 \frac{T(a_u(1-1/T(a_u)))}{T(a_u)} \left(1-\frac{1}{T(a_u)}\right)^{C_1 T(a_u)} \int_{1-1/T(a_u)}^1 \frac{dt}{\sqrt{1-t^2}} \\
&\geqslant C_3 T(a_u)^{-1/2}.
\end{aligned}$$

Here we have used (2.3) and the quasi-monotonicity of T. So we have (2.4). Then (2.5) follows from the definition of T.

(c) We can assume $\beta > 1$. Then by (2.5), and quasi-monotonicity of T,

$$C_1 \leqslant \frac{T(a_{\beta u})}{T(a_u)} \sim \left[\frac{\beta u}{Q(a_{\beta u})}\right]^2 \Big/ \left[\frac{u}{Q(a_u)}\right]^2 \leqslant \beta^2.$$

(d) Now

$$\begin{aligned}
\alpha u &= \frac{2}{\pi}\int_0^1 a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-t^2}} \geqslant \frac{2}{\pi}\int_{a_u/a_{\alpha u}}^1 a_u Q'(a_u) \frac{dt}{\sqrt{1-t^2}} \\
&\geqslant C_2 u T(a_u)^{1/2} \left(1-\frac{a_u}{a_{\alpha u}}\right)^{1/2}
\end{aligned}$$

by (2.4). Hence

$$1-\frac{a_u}{a_{\alpha u}} \leqslant C_3/T(a_u).$$

In the other direction,

$$\begin{aligned}
\alpha u &= \frac{2}{\pi}\left[\int_0^{a_u/a_{\alpha u}} + \int_{a_u/a_{\alpha u}}^1\right] a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-t^2}} \\
&\leqslant \frac{2}{\pi}\int_0^{a_u/a_{\alpha u}} a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-(a_{\alpha u} t/a_u)^2}} + \frac{2}{\pi} a_{\alpha u} Q'(a_{\alpha u}) \int_{a_u/a_{\alpha u}}^1 \frac{dt}{\sqrt{1-t}} \\
&\leqslant \frac{a_u}{a_{\alpha u}}\left[\frac{2}{\pi}\int_0^1 a_u s Q'(a_u s) \frac{ds}{\sqrt{1-s^2}}\right] + \frac{4}{\pi} a_{\alpha u} Q'(a_{\alpha u}) \left(1-\frac{a_u}{a_{\alpha u}}\right)^{1/2} \\
&\leqslant u + CuT(a_u)^{1/2}\left(1-\frac{a_u}{a_{\alpha u}}\right)^{1/2}
\end{aligned}$$

by (2.4) and (2.6). Then

$$1-\frac{a_u}{a_{\alpha u}} \geqslant \left(\frac{\alpha-1}{C}\right)^2 \frac{1}{T(a_u)}.$$

(e) We apply (1.5) with $y=a_u$ and $x=C_3$, so that

$$a_u Q'(a_u) \leqslant C_4 Q(a_u)^{C_2}$$
$$\Rightarrow uT(a_u)^{1/2} \leqslant C_5(uT(a_u)^{-1/2})^{C_2}.$$

Rearranging this gives (2.8).

(f) For large enough u,

$$\frac{Q(a_{\alpha u})}{Q(a_u)} = \exp\left(\int_{a_u}^{a_{\alpha u}} \frac{T(t)}{t}\,dt\right)$$
$$\geqslant \exp\left(C_6 T(a_u) \log\left(\frac{a_{\alpha u}}{a_u}\right)\right) \geqslant \exp(C_7) > 1,$$

by (d) of this lemma.

(g) From (1.5) with $y=a_{Lu}$ and $x=a_u$,

$$\frac{T(a_{Lu})}{T(a_u)} \leqslant C\left(\frac{Q(a_{Lu})}{Q(a_u)}\right)^{C_2-1}.$$

This forces $C_2>1$, as the left-hand side $\to\infty$ as $L\to\infty$. Then with the constants in $\sim$ independent of L, (2.5) gives

$$\frac{Q(a_{Lu})}{Q(a_u)} \sim \frac{LuT(a_{Lu})^{-1/2}}{uT(a_u)^{-1/2}}$$
$$\geqslant CL\left(\frac{Q(a_{Lu})}{Q(a_u)}\right)^{-(C_2-1)/2}$$
$$\Rightarrow \frac{Q(a_{Lu})}{Q(a_u)} \geqslant CL^{2/(1+C_2)}.$$

Then using (2.1),

$$\left(\frac{a_{Lu}}{a_u}\right)^{C_1 T(a_{Lu})} \geqslant CL^{2/(1+C_2)}.$$

We deduce the right-hand inequality in (2.10) from this last inequality and the inequality $\log t \leqslant t-1$, $t\geqslant 1$. In the other direction, (2.1) and then (2.5) give

$$\frac{a_{Lu}}{a_u} \leqslant \left(\frac{Q(a_{Lu})}{Q(a_u)}\right)^{1/(C_2 T(a_u))}$$

$$\leqslant \left(C_1 \frac{LuT(a_{Lu})^{-1/2}}{uT(a_u)^{-1/2}}\right)^{1/(C_2 T(a_u))} \leqslant (C_3 L)^{1/(C_2 T(a_u))}.$$

Here the constants are independent of L and u. Then the left inequality in (2.10) follows. ▌

We finish this section with an infinite finite-range inequality: We provide a proof because those in the literature [13, 18, 20, ...], don't quite match our needs/hypotheses:

Lemma 2.3. *Let $0 < p \leqslant \infty$, $s > 1$. Then for some L, C_1, $C_2 > 0$, $n \geqslant 1$, and $P \in \mathscr{P}_n$,*

$$\|PW\|_{L_p(\mathbb{R})} \leqslant C_1 \, \|PW\|_{L_p(-a_{sn},\, a_{sn})}. \tag{2.11}$$

Moreover,

$$\|PW\|_{L_p(|x| \geqslant a_{sn})} \leqslant C_1 e^{-C_2 n T(a_n)^{-1/2}} \, \|PW\|_{L_p(-a_{sn},\, a_{sn})}. \tag{2.12}$$

Remark. Note that (2.8) of Lemma 2.2(e) shows that for some $C_3 > 0$, and large enough n,

$$nT(a_n)^{-1/2} \geqslant n^{C_3}.$$

Proof. We may change Q in a finite interval without affecting (2.11), (2.12) apart from increasing the constants. Note too that the affect on a_u is marginal, and is absorbed into the fact that $s > 1$. Thus we may assume that Q' is continuous in $[-1, 1]$. This and the strict monotonicity of $tQ'(t)$ in $(0, \infty)$, allow us to apply existing sup-norm inequalities to deduce that for $P \in \mathscr{P}_n$,

$$\|PW\|_{L_\infty(\mathbb{R})} \leqslant C \, \|PW\|_{L_\infty[-a_{sn},\, a_{sn}]}.$$

For a precise reference, see [25; 9, Theorem 4.5]. Moreover, the proof of Lemma 5.1 in [13, pp.231–232] gives without change for $p < \infty$

$$|PW|^p \, (a_n x) \leqslant \frac{1}{\pi} \frac{2x}{x-1} \int_{-1}^{1} |PW|^p \, (a_n t) \, dt, \qquad x > 1. \tag{2.13}$$

Let $\langle x\rangle$ denote the greatest integer $\leqslant x$. Let δ be small and positive, let $l:=\langle \delta n\rangle$ and let $T_l(x)$ denote the Chebyshev polynomial of degree l. Using the identity

$$T_l(x)=\tfrac{1}{2}[(x+\sqrt{x^2-1})^l+(x-\sqrt{x^2-1})^l], \qquad x>1, \tag{2.14}$$

it is not difficult to see that

$$T_l(x)\geqslant\left\{\begin{matrix} \frac{1}{2}\exp\left(\frac{l}{\sqrt{2}}\sqrt{x-1}\right), & x\in\left(1,\frac{9}{8}\right) \\ \frac{1}{2}x^l, & x\geqslant 1 \end{matrix}\right\}. \tag{2.15}$$

We now let $m:=n+l=n+\langle \delta n\rangle$, $m':=n+2l=n+2\langle \delta n\rangle$ and apply (2.13) to $P(x)\,T_l(x/a_m)\in\mathcal{P}_m$. We obtain for $x>1$,

$$|PW|^p\,(a_m x)\leqslant T_l(x)^{-p}\frac{1}{\pi}\frac{2x}{x-1}\int_{-1}^{1}|PW|^p\,(a_m t)\,dt.$$

Replacing $a_m x$ by y, and integrating from $a_{m'}$ gives

$$\int_{a_{m'}}^{\infty}|PW|^p\,(y)\,dy\leqslant\left(\int_{-a_m}^{a_m}|PW|^p\,(s)\,ds\right)\left(\frac{2}{\pi}\int_{a_{m'}}^{\infty}\frac{y}{y-a_m}\,T_l\left(\frac{y}{a_m}\right)^{-p}\frac{dy}{a_m}\right).$$

Here using (2.15),

$$\begin{aligned}
&\int_{a_{m'}}^{\infty}\frac{y}{y-a_m}\,T_l\left(\frac{y}{a_m}\right)^{-p}\frac{dy}{a_m}\\
&\quad=\int_{a_{m'}/a_m}^{\infty}\frac{x}{x-1}\,T_l(x)^{-p}\,dx\\
&\quad\leqslant C\left(\int_{a_{m'}/a_m}^{9/8}\frac{1}{x-1}\exp\left(-\frac{lp}{\sqrt{2}}\sqrt{x-1}\right)dx+\int_{9/8}^{\infty}x^{-lp}\,dx\right)\\
&\quad\leqslant C_1\left(\log\left(\frac{8}{(a_{m'}/a_m)-1}\right)\exp\left(-C_2 l\left(\frac{a_{m'}}{a_m}-1\right)^{1/2}\right)+\left(\frac{9}{8}\right)^{-lp}\right)\\
&\quad\leqslant C_3\exp(-C_4 nT(a_n)^{-1/2}).
\end{aligned}$$

Here we have used (2.7) and our choice of l. Now if δ is small enough, $m'\leqslant sn$. Then (2.12) follows easily, and in turn yields (2.11). The proof for $p=\infty$ is similar but easier. ∎

3. TECHNICAL LEMMAS ON Φ_t

In this section, we present various estimates involving the functions σ and Φ_t. Throughout, we assume that $W=e^{-Q}\in\mathscr{E}_1$. Recall that

$$\sigma(t):=\inf\left\{a_u : \frac{a_u}{u}\leqslant t\right\}, \qquad t>0;$$

$$\Phi_t(x)=\sqrt{\left|1-\frac{|x|}{\sigma(t)}\right|}+T(\sigma(t))^{-1/2}, \qquad x>0.$$

LEMMA 3.1. (a) *There exists* s_0, v_0 *such that for* $s\in(0,s_0)$, *we can write* $s=a_v/v$, *where* $v\geqslant v_0$. *Moreover, we can write*

$$\sigma(s)=\sigma\left(\frac{a_v}{v}\right)=a_{\beta(v)}, \tag{3.1}$$

where

$$1\geqslant\sigma\left(\frac{a_v}{v}\right)\Big/a_v=a_{\beta(v)}/a_v\geqslant 1-C/T(a_v). \tag{3.2}$$

In particular,

$$\lim_{v\to\infty}\frac{\beta(v)}{v}=1. \tag{3.3}$$

(b) *There exist* C_1, $C_2>0$ *such that for* $s/2\leqslant t\leqslant s$, *and* $s\leqslant C_1$,

$$1\leqslant\frac{\sigma(t)}{\sigma(s)}\leqslant 1+\frac{C_2}{T(\sigma(t))}. \tag{3.4}$$

(c) *There exist* C_1, C_2 *independent of* s, t, x, *such that for* $0<t<s\leqslant C_1$,

$$\Phi_s(x)\leqslant C_2\Phi_t(x), \qquad |x|\leqslant\sigma(s). \tag{3.5}$$

(d) *There exists* C_1, *such that for* $0<s\leqslant C_1$, *and* $s/2\leqslant t\leqslant s$,

$$\Phi_s(x)\sim\Phi_t(x), \qquad x\in\mathbb{R}. \tag{3.6}$$

(e) *Uniformly for* $x\in\mathbb{R}$, *and* $n\geqslant 1$,

$$\Phi_{a_n/n}(x)\sim\sqrt{\left|1-\frac{|x|}{a_n}\right|}+T(a_n)^{-1/2}. \tag{3.7}$$

Proof. (a) The existence of v for the given s follows from the fact that $u \to a_u$ is continuous and

$$\frac{a_u}{u} \to 0, \qquad u \to \infty.$$

The latter in turn follows from the faster than polynomial growth of Q and (2.5), which implies $Q(a_u) = o(u)$. The continuity of a_u allows us to write $\sigma(s) = a_{\beta(v)}$, some $\beta(v)$. Since

$$\sigma(s) = \sigma\left(\frac{a_v}{v}\right) \leqslant a_v$$

the left inequality in (3.2) follows. For the other direction, we note that by definition of $\sigma(a_v/v)$ and $\beta(v)$, we have $\beta(v) \leqslant v$ and

$$\frac{a_{\beta(v)}}{\beta(v)} \leqslant \frac{a_v}{v}$$

so

$$1 \leqslant \frac{v}{\beta(v)} \leqslant \frac{a_v}{a_{\beta(v)}} \leqslant \left(\frac{Q(a_v)}{Q(a_{\beta(v)})}\right)^{1/2}$$

for large enough v, by (2.1). Using (2.5), we obtain

$$1 \leqslant \frac{v}{\beta(v)} \leqslant C\left(\frac{vT(a_v)^{-1/2}}{\beta(v)\, T(a_{\beta(v)})^{-1/2}}\right)^{1/2} \leqslant C_1\left(\frac{v}{\beta(v)}\right)^{1/2}.$$

It follows that $v \leqslant C_2\beta(v)$ and so $v \sim \beta(v)$. Then

$$1 \leqslant \frac{v}{\beta(v)} \leqslant \frac{a_v}{a_{\beta(v)}} \to 1, \qquad v \to \infty,$$

by (2.7), so we have (3.3). Then (2.7) also gives the right inequality in (3.2).

(b) Write $s = a_u/u$ and $t = a_v/v$. Then as σ is decreasing,

$$1 \geqslant \frac{\sigma(s)}{\sigma(t)} = \frac{a_{\beta(u)}}{a_{\beta(v)}}.$$

If we can show that

$$u \sim v \tag{3.8}$$

which in turn implies that

$$\beta(u) \sim \beta(v),$$

then (2.7) gives

$$1 \geqslant \frac{\sigma(s)}{\sigma(t)} \geqslant 1 - \frac{C}{T(a_v)}$$

which together with (2.6) gives the result. We proceed to establish (3.8). Suppose that it is not true, say, for example, we can have

$$\frac{u}{v} \to \infty.$$

For the corresponding s, t, our hypothesis is

$$\frac{1}{2} \leqslant \frac{t}{s} = \frac{a_v}{a_u}\frac{u}{v} \leqslant 1.$$

Then

$$\frac{a_v}{a_u} \to 0 \tag{3.9}$$

and (2.1) gives

$$\frac{Q(a_u)}{Q(a_v)} \geqslant \left(\frac{a_u}{a_v}\right)^{C_2 T(a_v)} \geqslant \left(\frac{a_u}{a_v}\right)^2,$$

for large u, v. But from (2.5),

$$\left(\frac{a_u}{a_v}\right)^2 \leqslant \frac{Q(a_u)}{Q(a_v)} \sim \frac{uT(a_u)^{-1/2}}{vT(a_v)^{-1/2}} \leqslant C\frac{u}{v} \leqslant C\frac{a_u}{a_v},$$

again by our hypotheses on s, t. This contradicts (3.9). So we have (3.8) and the result.

(c) Let $\delta > 0$ be fixed. Firstly for $1 - |x|/\sigma(s) \geqslant \delta/T(\sigma(s))$,

$$\Phi_s(x) \sim \sqrt{1 - \frac{|x|}{\sigma(s)}} \leqslant \sqrt{1 - \frac{|x|}{\sigma(t)}} \leqslant \Phi_t(x).$$

Next, for $|1 - |x|/\sigma(s)| \leqslant \delta/T(\sigma(s))$,

$$\Phi_s(x) \sim T(\sigma(s))^{-1/2}.$$

This is bounded by $C\Phi_t(x)$ if $|1-|x|/\sigma(t)|\geqslant\delta/T(\sigma(s))$, for a fixed $\delta>0$. Otherwise, we have $|1-|x|/\sigma(s)|\leqslant\delta/T(\sigma(s))$ and $|1-|x|/\sigma(t)|\leqslant\delta/T(\sigma(s))$, so

$$\left|1-\frac{\sigma(t)}{\sigma(s)}\right|=\left|\left(1-\frac{|x|}{\sigma(s)}\right)-\frac{|x|}{\sigma(s)}\left(\frac{\sigma(t)}{|x|}-1\right)\right|$$
$$\leqslant C_1\delta/T(\sigma(s)).$$

If δ is small enough, we deduce from (2.7) and (2.6) that

$$T(\sigma(t))\sim T(\sigma(s))$$

and again (3.5) follows.

(d) Write $s=a_u/u$ and $t=a_v/v$. Then we have (3.8), so

$$\left|1-\frac{|x|}{\sigma(t)}\right|=\left|1-\frac{|x|}{\sigma(s)}+\left[\frac{|x|}{\sigma(s)}-1+1\right]\left(1-\frac{\sigma(s)}{\sigma(t)}\right)\right|$$
$$\leqslant\left|1-\frac{|x|}{\sigma(s)}\right|\left[1+O\left(\frac{1}{T(\sigma(s))}\right)\right]+O\left(\frac{1}{T(\sigma(s))}\right).$$

Then we obtain for $x\in\mathbb{R}$,

$$\left|1-\frac{|x|}{\sigma(t)}\right|^{1/2}\leqslant C\Phi_s(x).$$

Also $T(\sigma(t))\sim T(\sigma(s))$, so

$$\Phi_t(x)\leqslant C\Phi_s(x).$$

The converse inequality follows similarly.

(e) By (a), we can write

$$\sigma\left(\frac{a_n}{n}\right)=a_{\beta(n)}=a_{n(1+o(1))}.$$

Recall that

$$\Phi_{a_n/n}(x)=\sqrt{\left|1-\frac{|x|}{\sigma(a_n/n)}\right|+T\left(\sigma\left(\frac{a_n}{n}\right)\right)^{-1/2}}.$$

Here by (2.6) and (a) of this lemma,

$$T\left(\sigma\left(\frac{a_n}{n}\right)\right)\sim T(a_n)$$

and much as in (d),

$$\left|1-\frac{|x|}{\sigma(a_n/n)}\right| \sim \left|1-\frac{|x|}{a_n}\right|$$

for large n and $|x| \leqslant a_{n/2}$ or $|x| \geqslant a_{2n}$. In the range $a_{n/2} \leqslant |x| \leqslant a_{2n}$, both the left- and right-hand sides of (3.7) are $\sim T(a_n)^{-1/2}$. ∎

LEMMA 3.2. (a) *Let $L > 0$. Uniformly for $u \geqslant 1$, and $|x|, |y| \leqslant a_u$, such that*

$$|x-y| \leqslant L\frac{a_u}{u}\sqrt{\left|1-\frac{|y|}{a_u}\right|}, \tag{3.10}$$

we have

$$W(x) \sim W(y) \tag{3.11}$$

and

$$1-\frac{|x|}{a_{2u}} \sim 1-\frac{|y|}{a_{2u}}. \tag{3.12}$$

(b) *Let $L, M > 0$. For $t \in (0, t_0)$, $|x|, |y| \leqslant \sigma(Mt)$ such that*

$$|x-y| \leqslant Lt\Phi_t(x), \tag{3.13}$$

we have (3.11) and

$$\Phi_t(x) \sim \Phi_t(y). \tag{3.14}$$

Proof. (a) It suffices to prove (3.11), (3.12) for large u. Moreover, (3.11) and (3.12) are immediate for $|x| \leqslant C$, and large u. Let us suppose that $C \leqslant x \leqslant y \leqslant x + L(a_u/u)\sqrt{|1-|y|/a_u|}$. Then as $Q'(s)$ is quasi-increasing for large s,

$$0 \leqslant Q(y)-Q(x) \leqslant C_1 Q'(y)(y-x).$$

We have then (3.11) for

$$y-x = O\left(\frac{1}{Q'(y)}\right). \tag{3.15}$$

We shall show that

$$a_u Q'(y)\sqrt{\left|1-\frac{y}{a_u}\right|} \leqslant C_2 u, \tag{3.16}$$

so that (3.10) implies (3.15) and hence (3.11). If firstly, $0 < y \leqslant a_u/2$, then

$$\begin{aligned} a_u Q'(y)\sqrt{\left|1-\frac{y}{a_u}\right|} &\leqslant C_3 a_u Q'(y) \int_{1/2}^{1} \frac{dt}{\sqrt{1-t^2}} \\ &\leqslant C_4 \int_{1/2}^{1} a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}} \leqslant C_5 u. \end{aligned}$$

If on the other hand, $a_u/2 \leqslant y \leqslant a_u$,

$$a_u Q'(y)\sqrt{\left|1-\frac{y}{a_u}\right|} \leqslant C_6 \int_{y/a_u}^{1} a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}} \leqslant C_7 u.$$

So we have (3.16) in all cases. Next from (3.10) and as $y \leqslant a_u$,

$$\begin{aligned} 1 \leqslant \frac{1-x/a_{2u}}{1-y/a_{2u}} &= 1 + \frac{y-x}{a_{2u}(1-y/a_{2u})} = 1 + O\left(\frac{1}{u\sqrt{1-y/a_{2u}}}\right) \\ &= 1 + O\left(\frac{1}{u\sqrt{1-a_u/a_{2u}}}\right) = 1 + O\left(\frac{T(a_u)^{1/2}}{u}\right) = 1 + o(1), \end{aligned}$$

by (2.7) and (2.8).

(b) Write $Mt = a_u/u$, so that $|x|, |y| \leqslant \sigma(Mt) \leqslant a_u$, and we can recast (3.13) as

$$|x-y| \leqslant C_1 \frac{a_u}{u}\left[\sqrt{1-\frac{|x|}{a_u}} + T(a_u)^{-1/2}\right] \leqslant C_2 \frac{a_{2u}}{2u}\sqrt{1-\frac{|x|}{a_{2u}}}$$

by (2.7), (3.6), and (3.7). Then (a) gives (3.11), and (3.14) follows easily from (3.12). ▮

4. POLYNOMIAL APPROXIMATION OF W^{-1}

The result of this section is:

THEOREM 4.1. *For $n \geqslant 1$, there exist polynomials G_n of degree at most Cn, such that*

$$0 \leqslant G_n(x) \leqslant W^{-1}(x), \qquad x \in \mathbb{R}; \tag{4.1}$$

and

$$G_n(x) \sim W^{-1}(x), \qquad |x| \leqslant a_n. \tag{4.2}$$

We remark that this does not follow from existing results in the literature on approximation by weighted polynomials of the form $P_n(x)\, W(a_n x)$ [14, 26] as our weights do not satisfy their hypotheses. The methods of Totik [26] can be applied to give sharper results but we base our proof on:

LEMMA 4.2. *There exists an even entire function*

$$G(x) = \sum_{j=0}^{\infty} g_j x^{2j}, \qquad g_j \geqslant 0 \quad \forall j, \tag{4.3}$$

such that

$$G(x) \sim W^{-1}(x), \qquad x \in \mathbb{R}. \tag{4.4}$$

Proof. Set

$$Q_1(r) := Q(\sqrt{r});$$

and

$$\psi(r) := rQ_1'(r) = \tfrac{1}{2}\sqrt{r}\, Q'(\sqrt{r}).$$

Then ψ is increasing in $(0, \infty)$, and if $\lambda > 1$, $r \geqslant r_0$, the quasi-increasing nature of Q' gives for some $C \neq C(\lambda)$,

$$\psi(\lambda r) - \psi(r) \geqslant \tfrac{1}{2}\sqrt{r}\, Q'(\sqrt{r})\, (\sqrt{\lambda}\, C - 1) \geqslant 1$$

if λ is large enough. Moreover, $\phi(r) := e^{Q_1(r)}$ admits the representation

$$\phi(r) = \phi(1) \exp\left(\int_1^r \frac{\psi(s)}{s}\, ds\right), \qquad r \geqslant 1.$$

By a theorem of Clunie and Kövari [2, Theorem 4, p. 19], there exists entire

$$G_1(r) = \sum_{j=0}^{\infty} g_j r^j, \qquad g_j \geqslant 0 \quad \forall j$$

such that

$$G_1(r) \sim \phi(r) := \exp(Q(\sqrt{r})), \qquad r \geqslant r_0.$$

Then assuming $g_0 > 0$ as we can, we see that

$$G(r) := G_1(r^2)$$

satisfies (4.4). ∎

In the analogous construction for Freud weights, the second author and Z. Ditzian used as the polynomials G_n the partial sums of G. However, in the Erdős case, for partials sums of degree $O(n)$, we only have

$$G_n(x) \sim W^{-1}(x)$$

for $|x| \leqslant q_n$, where q_n is Freud's quantity, the root of the equation

$$n = q_n Q'(q_n).$$

Although $a_n/q_n \to 1$, $n \to \infty$ for Erdős weights, in effect, q_n is significantly smaller than a_n. (We cannot properly describe, using only q_n, the improvement in the degree of approximation near $\pm a_n$.) So we use a more sophisticated interpolant:

Proof of Theorem 4.1. Let J be a positive even integer (to be chosen large enough later) and let $T_n(x)$ denote the classical Chebyshev polynomial on $[-1, 1]$. Let G_n denote the Lagrange interpolant to G at the zeros of $T_n(x/a_n)^J$ so that G_n has degree at most $Jn-1$, and admits the error representation

$$(G - G_n)(x) = \frac{1}{2\pi i} \int_\Gamma \frac{G(t)}{t-x} \left(\frac{T_n(x/a_n)}{T_n(t/a_n)} \right)^J dt$$

for x inside Γ. We shall choose Γ to be the ellipse with foci at $\pm a_n$, intersecting the real and imaginary axes at $(a_n/2)(\rho + \rho^{-1})$ and $(a_n/2)(\rho - \rho^{-1})$, respectively. Here we shall choose for some fixed small $\varepsilon > 0$,

$$\rho := 1 + \left(\frac{\varepsilon}{T(a_n)} \right)^{1/2}.$$

Since G has non-negative Maclaurin series coefficients, and satisfies (4.4), we deduce that

$$\begin{aligned} \delta_n &:= \|G_n/G - 1\|_{L_\infty[-a_n, a_n]} \\ &\leqslant C_1 \frac{W^{-1}((a_n/2)(\rho + \rho^{-1}))}{(\rho - 1)^2} \frac{1}{\min_{t \in \Gamma} |T_n(t/a_n)|^J}. \end{aligned}$$

Now for $t\in\Gamma$, we can write $t=(a_n/2)(z+z^{-1})$ where $|z|=\rho$, so that

$$|T_n(t/a_n)|=|T_n(\tfrac{1}{2}(z+z^{-1}))|=|\tfrac{1}{2}(z^n+z^{-n})|$$
$$\geqslant\tfrac{1}{2}(\rho^n-\rho^{-n})\geqslant\exp(C_2 nT(a_n)^{-1/2}).$$

(Recall that $nT(a_n)^{-1/2}\to\infty$ as $n\to\infty$ and in fact grows faster than a power of n.) It is important here that C_2 is independent of J. Next

$$\frac{a_n}{2}(\rho+\rho^{-1})\leqslant a_n\left(1+C_3\frac{\varepsilon}{T(a_n)}\right)\leqslant a_{2n}$$

if ε is small enough, and n is large enough, by (2.7). Then

$$W^{-1}\left(\frac{a_n}{2}(\rho+\rho^{-1})\right)\leqslant\exp(C_4 Q(a_{2n}))\leqslant\exp(C_5 nT(a_n)^{-1/2}),$$

where again it is important that C_5 is independent of J. Since $(\rho-1)^{-2}\sim T(a_n)$ grows no faster than a power of n, we see that choosing J large enough, gives

$$\delta_n\to 0,\qquad n\to\infty.$$

Then (4.4) gives (4.2).

We now turn to proving (4.1). It suffices to prove

$$0\leqslant G_n\leqslant CW^{-1}$$

for then (4.1) follows on multiplying G_n by a suitable constant. Firstly, we can assume n is even (for odd n, we can use G_{n+1}) so that $H_n(x):=G_n(\sqrt{x})$ is a polynomial of degree at most $Jn/2-1$ (recall T_n and J are even) that interpolates to the entire function $H(x):=G(\sqrt{x})$ at the $Jn/2$ zeros of $T_n(\sqrt{t}/a_n)^J$ that lie in $(0, a_n^2)$. Thus $H_n(x)$ is determined entirely by interpolation conditions. Let γ_n denote the leading coefficient of $T_n(x/\sqrt{a_n})$. Then the usual derivative-error formula for Hermite interpolation gives for $x\in(0,\infty)$ and some $\xi=\xi(x)\in(0,\infty)$,

$$(H-H_n)(x)=\gamma_n^{-J}T_n\left(\frac{\sqrt{x}}{a_n}\right)^J\frac{H^{(Jn/2)}(\xi)}{(Jn/2)!}\geqslant 0.$$

(Recall that H is entire and has non-negative Maclaurin series coefficients.) So in $\mathbb{R}$

$$G_n\leqslant G\leqslant CW^{-1}.$$

To show that $G_n \geqslant 0$ in $\mathbb{R}$, we note that it is true in $[-a_n, a_n]$ and we must establish it elsewhere. We use an idea employed in the Posse–Markov–Stieltjes inequalities [8, p. 30, Lemma 5.3] (there the proof is for $(-\infty, \infty)$, but the proof goes through for $(0, \infty)$ with trivial changes). Now H is absolutely monotone in $(0, \infty)$ and $H - H_n$ has $Jn/2$ zeros in $(0, a_n^2]$. If m is the number of zeros of $H_n(x)$ in $[a_n^2, \infty)$, Lemma 5.3 in [8, p. 30] gives

$$\frac{Jn}{2} + m \leqslant \deg(H_n) + 1 \leqslant \frac{Jn}{2}.$$

So $m = 0$, that is, H_n has no zeros in (a_n^2, ∞). Thus $H_n \geqslant 0$ there, so $G_n \geqslant 0$ in $\mathbb{R}$. ∎

5. POLYNOMIALS APPROXIMATING CHARACTERISTIC FUNCTIONS

Our Jackson theorem is based on polynomial approximations to the characteristic function $\chi_{[a, b]}$ of an interval $[a, b]$. We believe the following result is of independent interest:

THEOREM 5.1. *Let l be a positive integer. There exist J, C_1, n_0 such that for $n \geqslant n_0$ and $\tau \in [-a_n, a_n]$, there exist polynomials $R_{n, \tau}$ of degree at most $2lJn$ such that for $x \in \mathbb{R}$,*

$$|\chi_{[\tau, a_n]} - R_{n, \tau}|\,(x)\; W(x)/W(\tau) \leqslant C_1 \left(1 + \frac{n\,|x - \tau|}{a_n \sqrt{1 - |\tau|/a_{2n}}}\right)^{-l}. \tag{5.1}$$

We emphasize that the constants J, C_1, n_0 are independent of n, τ, x.

Remark. The method of proof of Theorem 5.1 in the unweighted case goes back to an old paper of Brudnyi [1]. We also make heavy use of polynomials from [12] built on the Chebyshev polynomials.

LEMMA 5.2. *There exist C_1, B, n_1 such that for $n \geqslant n_1$ and $|\zeta| \leqslant \cos \pi/2n$, there exists a polynomial $V_{n, \zeta}$ of degree at most $n - 1$ with*

$$\|V_{n, \zeta}\|_{L_\infty[-1, 1]} = V_{n, \zeta}(\zeta) = 1; \tag{5.2}$$

$$|V_{n, \zeta}(t)| \leqslant \frac{B\sqrt{1 - |\zeta|}}{n\,|t - \zeta|}, \qquad t \in (-1, 1) \backslash \{\zeta\}. \tag{5.3}$$

Moreover,

$$V_{n,\zeta}(t) \geq \frac{1}{2}, \qquad |t-\zeta| \leq C_1 \frac{\sqrt{1-|\zeta|}}{n}. \tag{5.4}$$

The constants are independent of n, ζ, t.

Proof. The assertions (5.2), (5.3) are Proposition 13.1 in [12]. The estimate (5.4) follows from the classical Bernstein inequality. ∎

The polynomials $R_{n,\tau}$ are determined as follows: Let us suppose that, say,

$$a_1 \leq \tau \leq a_n.$$

Later on, we shall suppose that τ exceeds a fixed positive constant. We define

$$\zeta := \frac{\tau}{a_{2lJn}} \tag{5.5}$$

and if the G_n are the polynomials of Theorem 4.1,

$$R_{n,\tau}(x) := \frac{\int_0^x G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{\int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}. \tag{5.6}$$

The parameters $\tau^* > \tau$ and J are defined as follows: Let $A \in (0, 1]$ denote the constant in the quasi-monotonicity of Q', so that

$$Q'(y) \geq AQ'(x), \qquad y \geq x \geq 1. \tag{5.7}$$

Let M denote a positive constant such that for say, $u \geq u_0$,

$$Q'(x) \leq MQ'(a_u), \qquad 1 \leq x \leq a_{2u}. \tag{5.8}$$

The existence of such an M follows from (2.4), (2.6). We set

$$H := H(n, \tau, l) := \frac{2l\,n}{Aa_n Q'(\tau)\sqrt{1-\zeta}} \tag{5.9}$$

and if $\tau = a_r$,

$$\tau^* := \tau^*(n, \tau) := \min\left\{a_{2r}, a_n, \tau + 2\frac{a_n}{n}\sqrt{1-\zeta}\, H \log H\right\}. \tag{5.10}$$

The reason for this (complicated!) choice will become clearer later. We assume that $J \geqslant 4$ is so large that G_n has degree at most $Jn-1$, and also

$$J \geqslant 16M/A, \tag{5.11}$$

where A, M are as above. We also assume that J is a multiple of 4. Note that then $R_{n,\tau}$ has degree at most $Jn + lJn$. We first record some estimates of the terms in (5.6):

LEMMA 5.3. (a) *For $n \geqslant n_1$, and $C_1 \leqslant \tau \leqslant a_n$, we have*

$$W(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds \geqslant C_2 \frac{a_n}{n} \sqrt{1-\zeta}, \tag{5.12}$$

where $C_2 \neq C_2(n, \tau)$.

(b) *For $x \in (\tau, a_{2lJn})$,*

$$\int_x^{a_{2lJn}} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ/2} ds \leqslant C_1 \frac{a_n}{n} \sqrt{1-\zeta}\left(1 + \frac{n\,|x-\tau|}{a_n\sqrt{1-\zeta}}\right)^{-l} \tag{5.13}$$

and for $x \in (-a_{2lJn}, \tau)$,

$$\int_{-a_{2lJn}}^{x} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ/2} ds \leqslant C_1 \frac{a_n}{n} \sqrt{1-\zeta}\left(1 + \frac{n\,|x-\tau|}{a_n\sqrt{1-\zeta}}\right)^{-l}. \tag{5.14}$$

Here $C_1 \neq C_1(n, \tau)$.

Proof. (a) Let us denote the left-hand side of (5.12) by Γ. By (4.2) and (5.4),

$$\Gamma \geqslant C_2\, W(\tau) \int_{\tau - C_3(a_n/n)\sqrt{1-\zeta}}^{\tau} W^{-1}(s)\, ds \geqslant C_4 \frac{a_n}{n} \sqrt{1-\zeta},$$

where we have used (3.11) of Lemma 3.2(a).

(b) These follow in a straightforward fashion from the estimates (5.2), (5.3) and the fact that $J \geqslant 4$, so $lJ/2 > l+1$. ∎

Now we begin the proof of Theorem 5.1. We first show that it suffices to consider τ in the range $[S, a_n]$, for some fixed S.

Proof of Theorem 5.1 for $|\tau| \leqslant S$, where S is fixed. Note first that since for such τ,

$$W(x)/W(\tau) \leqslant W(0)/W(S), \qquad x \in \mathbb{R},$$

we must only prove there exists $R_{n,\tau}$ of degree at most n such that

$$|\chi_{[\tau, a_n]} - R_{n,\tau}|\,(x) \leqslant C_1 \left(1 + \frac{n\,|x-\tau|}{a_n\sqrt{1-|\tau|/a_{2n}}}\right)^{-l},$$

for $|x| \leqslant a_{2n}$, and then our infinite-finite range inequality Lemma 2.3 gives the rest. Setting here $\xi := \tau/a_n$, $s := x/a_n$, and $U_{n,\xi}(s) := R_{n,\tau}(x) = R_{n,\tau}(a_n s)$, we see that it suffices to show

$$|\chi_{[\xi, 1]}(s) - U_{n,\xi}(s)| \leqslant C_2(1 + n\,|s-\xi|)^{-l}, \qquad s \in [-2, 2].$$

We have used here that $|\xi| \leqslant \frac{1}{2}$, for large n. The existence of such polynomials is classical. See, for example, [4]. One could also base them on the $V_{n,\zeta}$ above. ∎

It suffices to consider $\tau \in [S, a_n]$, *where S is fixed.* Once this is done, we have the result for all $\tau \in [0, a_n]$. With the result for $\tau \geqslant 0$, we set

$$R_{n,-\tau}(x) := 1 - R_{n,\tau}(-x), \qquad x \in \mathbb{R}.$$

It is not difficult to check the result for $-\tau$ from the corresponding result for τ, using the identity

$$\chi_{[-\tau, a_n]}(x) = 1 - \chi_{(\tau, a_n]}(-x), \qquad x \in [a_{-n}, a_n]. \quad ∎$$

In the sequel, we define $R_{n,\tau}$ by (5.5)–(5.10).

It suffices to prove (5.1) *for* $\tau \in [S, a_n]$ *and* $|x| \leqslant a_{2lJn}$. Then (5.1) for this restricted range implies

$$\left\| \left(1 + \left[\frac{n(x-\tau)}{a_n\sqrt{1-\tau/a_{2n}}}\right]^2\right)^l R_{n,\tau}(x)\,\frac{W(x)}{W(\tau)} \right\|_{L_\infty[-a_{2lJn},\, a_{2lJn}]} \leqslant C_3 n^{C_4},$$

where $C_4 \neq C_4(n, \tau)$. Since the polynomial in the left-hand side has degree at most $2l + Jn + lJn \leqslant \eta 2lJn$, some fixed $\eta < 1$, if $l \geqslant 2$ and n is large enough (as we can assume), then the infinite-finite range inequality Lemma 2.3 gives

$$\left\| \left(1 + \left[\frac{n(x-\tau)}{a_n\sqrt{1-\tau/a_{2n}}}\right]^2\right)^l R_{n,\tau}(x)\,\frac{W(x)}{W(\tau)} \right\|_{L_\infty(|x| \geqslant a_{2lJn})} \leqslant C_5 \exp(-n^{C_6}).$$

Then (5.1) follows for $|x| \geqslant a_{2lJn}$. ∎

We can now begin the proof of (5.1) proper. We consider 5 different ranges of x: $[0, \tau)$, $[\tau, \tau^*]$, $(\tau^*, a_n]$, $(a_n, a_{2lJn}]$, $[-a_{2lJn}, 0)$. Moreover, we set

$$\Delta(x) := |\chi_{[\tau, a_n]} - R_{n,\tau}|\,(x)\,W(x)/W(\tau).$$

Proof of (5.1) *for* $x \in [0, \tau)$. Here using (4.1), and then (5.12),

$$\Delta(x) = \frac{W(x) \int_0^x G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{W(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}$$
$$\leqslant C \frac{W(x) \int_0^x W^{-1}(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{(a_n/n)\sqrt{1-\zeta}}$$
$$\leqslant C \frac{\int_0^x V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{(a_n/n)\sqrt{1-\zeta}}$$

by the monotonicity of W. Then (5.14) gives the result. ∎

Proof of (5.1) *for* $x \in [\tau, \tau^*]$. Here

$$\Delta(x) = \frac{W(x) \int_x^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{W(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}$$
$$\leqslant C \frac{\int_x^{\tau^*} \exp(Q(s) - Q(x))\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{(a_n/n)\sqrt{1-\zeta}}$$

by (4.1) and (5.12). Now for $s \in (x, \tau^*)$, the property (5.8) of Q' gives (recall $\tau^* \leqslant a_{2r}$)

$$Q(s) - Q(x) \leqslant MQ'(a_r)(s-x) \leqslant MQ'(\tau)(s-\tau).$$

Then using our bounds on $V_{n,\zeta}$ in (5.2), (5.3), we have

$$\Delta(x) \leqslant C_1 \frac{\int_x^{\tau^*} \exp(MQ'(\tau)(s-\tau)) \min\{1, Ba_{2lJn}\sqrt{1-\zeta}/(n(s-\tau))\}^{lJ}\, ds}{(a_{2lJn}/n)\sqrt{1-\zeta}}$$
$$= C_1 B \int_{n(x-\tau)/Ba_{2lJn}\sqrt{1-\zeta}}^{n(\tau^*-\tau)/Ba_{2lJn}\sqrt{1-\zeta}} \exp\left(\frac{a_{2lJn}}{a_n}\, \frac{2lMBu}{AH}\right) \min\left\{1, \frac{1}{u}\right\}^{lJ} du$$
$$\leqslant C_2 \int_{n(x-\tau)/Ba_{2lJn}\sqrt{1-\zeta}}^{(2/B)\, H \log H} g(u) \min\left\{1, \frac{1}{u}\right\}^{lJ/2} du$$

for say $n \geqslant n_1 = n_1(J, l)$ by (5.10), and where

$$g(u) := \exp\left(\frac{4lMBu}{AH}\right) \min\left\{1, \frac{1}{u}\right\}^{lJ/2}.$$

We claim that if J is large enough,

$$g(u) \leqslant C_3, \qquad u \in \left[0, \frac{2}{B} H \log H\right],$$

with C_3 independent of τ, n. Firstly we claim that if l is large enough,

$$H \geqslant e; \qquad H \geqslant e^{B/2} \tag{5.15}$$

uniformly for $\tau \in [S, a_n]$ and $n \geqslant n_0(J, l)$. First recall that B, J, A, M are independent of l (see (5.3), (5.7), (5.8), (5.11)). Then also from (3.16) for $\tau \in [S, a_n]$

$$a_n Q'(\tau) \sqrt{1 - \frac{\tau}{a_{2n}}} \leqslant Cn,$$

with $C \neq C(n, \tau, l)$. Then from (5.9),

$$H \geqslant \frac{2l}{AC} \left(\frac{1 - \tau/a_{2n}}{1 - \tau/a_{2lJn}} \right)^{1/2}.$$

Here for $n \geqslant n_0(J, l)$, using $1 - u \leqslant \log(1/u)$, $u \in (0, 1]$, we obtain

$$\begin{aligned} \frac{1 - \tau/a_{2lJn}}{1 - \tau/a_{2n}} &= 1 + \frac{\tau}{a_{2n}} \frac{1 - a_{2n}/a_{2lJn}}{1 - \tau/a_{2n}} \\ &\leqslant 1 + \frac{\log(a_{2lJn}/a_{2n})}{1 - a_n/a_{2n}} \leqslant 1 + C_1 \log(lJ), \end{aligned}$$

by (2.7) and the left inequality in (2.10). Thus for $n \geqslant n_0(J, l)$, uniformly for $\tau \in [S, a_n]$,

$$H \geqslant \frac{C_2 l}{\sqrt{\log lJ}}.$$

So (5.15) follows if we choose l large enough. Then

$$g(u) \leqslant \exp\left(\frac{4lMB}{Ae} \right), \qquad u \in (0, 1].$$

Next, by elementary calculus, g has at most one local extremum in $[1, \infty)$, and this is a minimum. Thus in any subinterval of $[1, \infty)$, g attains its maximum at the endpoints of that interval. In particular, we must only check that $g((2/B) H \log H)$ is bounded. Note that by (5.15), $(2/B) H \log H \geqslant e > 1$. So

$$g\left(\frac{2}{B} H \log H \right) = \exp\left(l \log H \left\{ \frac{8M}{A} - \frac{J}{2} \right\} - \frac{Jl}{2} \log\left[\frac{2}{B} \log H \right] \right) \leqslant 1$$

as $J \geqslant 16M/A$ (see (5.11)) and $H \geqslant e^{B/2}$. So we have

$$\Delta(x) \leqslant C_4 \int_{n(x-\tau)/Ba_{2lJn}\sqrt{1-\zeta}}^{\infty} \min\left\{1, \frac{1}{u}\right\}^{lJ/2} du$$

and then (5.1) follows as $J \geqslant 4$. ∎

Proof of (5.1) *for* $x \in (\tau^*, a_n]$. Here

$$\begin{aligned}
\Delta(x) &= \frac{W(x) \int_{\tau^*}^{x} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{W(\tau) \int_0^{\tau^*} G_n(s)\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds} \\
&\leqslant C_1 \frac{\int_{\tau^*}^{x} \exp(Q(s) - Q(x))\, V_{n,\zeta}(s/a_{2lJn})^{lJ}\, ds}{(a_n/n)\sqrt{1-\zeta}} \\
&\leqslant C_2 \frac{n}{a_n\sqrt{1-\zeta}} \left(e^{Q\left(\frac{\tau+x}{2}\right) - Q(x)} \int_{\tau^*}^{\frac{\tau+x}{2}} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds \right. \\
&\quad \left. + \int_{\frac{\tau+x}{2}}^{x} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds \right) \\
&\leqslant C_3 \left\{ e^{Q\left(\frac{\tau+x}{2}\right) - Q(x)} \left[1 + \frac{n(\tau^* - \tau)}{a_n\sqrt{1-\zeta}} \right]^{-l} \right. \\
&\quad \left. + \left[1 + \frac{n(x-\tau)}{a_n\sqrt{1-\zeta}} \right]^{-l} \right\}
\end{aligned} \tag{5.16}$$

by (5.3) and (5.13). Here if $\tau^* > (\tau + x)/2$, the first term in the last two lines can be dropped and we already have the desired estimate. In the contrary case, we must estimate the first term. We note that we can assume that $\tau^* < a_n$, for otherwise the current range of x is empty. We consider two subcases (recall the definition (5.10) of τ^*):

(I) $\tau^* = \tau + 2(a_n/n)\sqrt{1-\zeta}\, H \log H$. We shall show that

$$\Gamma := \frac{Q(x) - Q((\tau + x)/2)}{l \log(1 + n(x-\tau)/a_n\sqrt{1-\zeta})} \geqslant 1. \tag{5.17}$$

Then the first part of the first term in the right-hand side of (5.16) already gives the desired estimate; the second part of that first term can be bounded by 1. By quasi-monotonicity (5.7) of Q',

$$Q(x) - Q\left(\frac{\tau + x}{2}\right) \geqslant AQ'(\tau)\left(\frac{x-\tau}{2}\right).$$

Setting

$$u := \frac{n(x-\tau)}{a_n \sqrt{1-\zeta}},$$

we have

$$\Gamma \geqslant \frac{AQ'(\tau)(a_n/n)\sqrt{1-\zeta}\, u}{2l \log(1+u)} = \frac{u}{H \log(1+u)}.$$

(Recall that H was defined at (5.9)). But

$$u \geqslant \frac{n(\tau^* - \tau)}{a_n \sqrt{1-\zeta}} = 2H \log H.$$

Recall from (5.15) that $H \geqslant e$. Then since the function $u/\log(1+u)$ is increasing for $u \geqslant 2H \log H \geqslant e$, we obtain

$$\Gamma \geqslant \frac{2H \log H}{H \log(1+2H \log H)}.$$

Using the inequality $1+2t \log t \leqslant t^2$, $t \geqslant 2$, we have

$$\Gamma \geqslant \frac{2 \log H}{\log(H^2)} = 1.$$

So we have (5.17) and the result.

(II) $\tau^* = a_{2r}$. In this case, from (2.7),

$$\tau^* - \tau = a_{2r} - a_r \sim \frac{a_r}{T(a_r)} = \frac{\tau}{T(\tau)}.$$

Now if $\tau^* \leqslant x \leqslant \tau(1+(1/T(\tau)))$, then

$$x - \tau \sim \tau^* - \tau$$

and the second part of the first term in the right-hand side of (5.16) already gives the desired estimate (the first part of the first term can be estimated by 1). If $x > \tau(1+(1/T(\tau)))$, then

$$\frac{x}{((x+\tau)/2)} \geqslant 1 + \frac{1}{2T(\tau)+1} \geqslant 1 + \frac{1}{3T(\tau)}$$

for large τ, so from (2.1),

$$\frac{Q(x)}{Q((x+\tau)/2)} \geqslant \left(1+\frac{1}{3T(\tau)}\right)^{C_2 T\left(\frac{x+\tau}{2}\right)} \geqslant C_3 > 1.$$

(Recall that $(\frac{x+\tau}{2}) > \tau$.) Then

$$e^{Q\left(\frac{\tau+x}{2}\right)-Q(x)}\left[1+\frac{n(\tau^*-\tau)}{a_n\sqrt{1-\zeta}}\right]^{-l} \leqslant e^{-C_4 Q(x)}\left[1+\frac{C_5 n\tau}{a_n T(\tau)\sqrt{1-\zeta}}\right]^{-l}.$$

This will admit the desired estimate, namely

$$C_6\left[1+\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}\right]^{-l}$$

provided

$$e^{C_4 Q(x)/l}\frac{\tau}{T(\tau)} \geqslant C_7(x-\tau).$$

But

$$e^{C_4 Q(x)/l}\frac{\tau}{T(\tau)} \geqslant C_8\frac{e^{C_4 Q(x)/l}}{T(x)} \geqslant C_9 Q(x) \geqslant C_{10}x > C_{10}(x-\tau)$$

by (2.5), (2.8), and the faster than polynomial growth of Q, so we have the desired estimate. ∎

Proof of (5.1) *for* $x\in(a_n, a_{2lJn}]$. Here, much as in the previous range,

$$\begin{aligned}\Delta(x) &= \frac{W(x)\int_0^x G_n(s)\,V_{n,\zeta}(s/a_{2lJn})^{lJ}\,ds}{W(\tau)\int_0^{\tau^*} G_n(s)\,V_{n,\zeta}(s/a_{2lJn})^{lJ}\,ds}\\ &\leqslant C_2\frac{n}{a_n\sqrt{1-\zeta}}\left(e^{Q\left(\frac{\tau+x}{2}\right)-Q(x)}\int_0^{\frac{\tau+x}{2}} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds\right.\\ &\quad\left.+\int_{\frac{\tau+x}{2}}^{x} V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ} ds\right)\\ &\leqslant C_3\left\{e^{Q\left(\frac{\tau+x}{2}\right)-Q(x)}+\left[1+\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}\right]^{-l}\right\}.\end{aligned}$$

We must show that the first term on the last right-hand side admits a bound that is a constant multiple of the second term on the last right-hand side. Let us write $x=a_v$ (so $v\geqslant n$) and $(\tau+x)/2=a_u$ (so that $u<v$). If firstly $u\geqslant n/2$, then

$$Q(x)-Q\left(\frac{\tau+x}{2}\right)\geqslant C_4 Q'(a_{n/2})(x-\tau)\geqslant C_5\,\frac{n}{a_n}\,T(a_n)^{1/2}\,(x-\tau)$$

$$\geqslant C_6\,\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}\geqslant C_7 l\log\left(1+\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}\right)$$

by (2.4),(2.7). (Recall that $\zeta=\tau/a_{2lJn}$.) In this case the result follows. If $u<n/2$,

$$Q(x)-Q\left(\frac{\tau+x}{2}\right)\geqslant Q(a_n)-Q(a_{n/2})$$

$$\geqslant C_8\,Q(a_n)\geqslant C_9 nT(a_n)^{-1/2}\geqslant C_{10}n^{C_{11}}$$

by (2.5), (2.8). Since

$$\left[1+\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}\right]^{-l}\geqslant n^{-C_{11}}$$

the result again follows. ∎

Proof of (5.1) *for* $x\in[-a_{2lJn},0)$. Here using the evenness of W and (4.1), (5.12) as before gives

$$\Delta(x)=\frac{W(x)\int_x^0 G_n(s)\,V_{n,\zeta}(s/a_{2lJn})^{lJ}\,ds}{W(\tau)\int_0^{\tau^*}G_n(s)\,V_{n,\zeta}(s/a_{2lJn})^{lJ}\,ds}$$

$$\leqslant C_2\,\frac{n}{a_n\sqrt{1-\zeta}}\left(\int_x^{x/2}V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ}ds+e^{Q(x/2)-Q(x)}\int_{x/2}^0 V_{n,\zeta}\left(\frac{s}{a_{2lJn}}\right)^{lJ}ds\right)$$

$$\leqslant C_3\left\{\left[1+\frac{n\,|x/2-\tau|}{a_n\sqrt{1-\zeta}}\right]^{-l}+e^{Q(x/2)-Q(x)}\left[1+\frac{n\tau}{a_n\sqrt{1-\zeta}}\right]^{-l}\right\}.$$

Here $|x/2-\tau|=(|x|/2)+\tau\sim|x-\tau|$. Also, if $|x|\leqslant\tau$, then $\tau\sim\tau+|x|=|x-\tau|$. Otherwise (recall $\tau\geqslant S$), we have

$$e^{Q(x/2)-Q(x)}\leqslant e^{-C_4Q(x)}\leqslant e^{-C_5\,|x|}\leqslant(C_6\,|x|)^{-l}.$$

Again as $|x|\,\tau\geqslant C_8(\tau+|x|)=C_8\,|x-\tau|$, the result follows. ∎

6. THE PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. Recall that our moduli of continuity are

$$\omega_{r,p}(f, W, t) := \sup_{0<h\leqslant t} \|W\Delta^r_{h\Phi_t(x)}(f, x, \mathbb{R})\|_{L_p(|x|\leqslant\sigma(2t))} + \inf_{P\in\mathcal{P}_{r-1}} \|(f-P)W\|_{L(|x|\geqslant\sigma(4t))}$$

and

$$\bar{\omega}_{r,p}(f, W, t) := \left(\frac{1}{t}\int_0^t \|W\Delta^r_{h\Phi_t(x)}(f, x, \mathbb{R})\|^p_{L_p(|x|\leqslant\sigma(2t))}\, dh\right)^{1/p} + \inf_{P\in\mathcal{P}_{r-1}} \|(f-P)W\|_{L_p(|x|\geqslant\sigma(4t))},$$

where

$$\sigma(t) = \inf\left\{a_u : \frac{a_u}{u}\leqslant t\right\}.$$

We need further moduli of continuity. If I is an interval, and $f: I\to\mathbb{R}$, we define for $t>0$,

$$\Lambda_{r,p}(f, t, I) := \sup_{0<h\leqslant t}\left(\int_I |\Delta^r_h(f, x, I)|^p\, dx\right)^{1/p} \tag{6.1}$$

and its averaged cousin

$$\Omega_{r,p}(f, t, I) := \left(\frac{1}{t}\int_0^t\int_I |\Delta^r_s(f, x, I)|^p\, dx\, ds\right)^{1/p}. \tag{6.2}$$

Note that for some C_1, C_2 depending only on r and p (not on f, I, t)

$$C_1\leqslant \Lambda_{r,p}(f, t, I)/\Omega_{r,p}(f, t, I)\leqslant C_2. \tag{6.3}$$

It seems that (6.3) first appeared in [23]. See also [4; 24, p. 191].

For large enough n, we choose a partition

$$-a_n = \tau_{0n} < \tau_{1n} < \cdots < \tau_{nn} = a_n \tag{6.4}$$

such that if

$$I_{kn} := [\tau_{kn}, \tau_{k+1,n}], \qquad 0\leqslant k\leqslant n-1, \tag{6.5}$$

then uniformly in k and n,

$$|I_{kn}| \sim \frac{a_n}{n} \sqrt{1 - \frac{|\tau_{kn}|}{a_{2n}}}. \tag{6.6}$$

($|I|$ denotes the length of the interval I.) We also set $I_{nn} := \varnothing$. There are many ways to do this. For example, one can choose $\tau_{0,n} := -a_n$ and for $1 \leqslant k \leqslant n$, determine $\tau_{k,n}$ by

$$\frac{\int_{\tau_{k-1,n}}^{\tau_{k,n}} (1/\sqrt{1 - |s|/a_{2n}})\, ds}{\int_{-a_n}^{a_n} (1/\sqrt{1 - |s|/a_{2n}})\, ds} = \frac{1}{n}.$$

Let us set

$$I_n := [-a_n, a_n] = \bigcup_{k=0}^{n-1} I_{kn}, \tag{6.7}$$

$$\theta_{kn}(x) := \chi_{[\tau_{kn}, a_n]}(x) = \chi_{\bigcup_{i=k}^{n-1} I_{in}}(x), \tag{6.8}$$

and

$$I_{kn}^* := I_{kn} \cup I_{k+1,n}, \qquad 0 \leqslant k \leqslant n-1. \tag{6.9}$$

By Whitney's theorem [24, p. 195], we can find a polynomial p_k of degree at most r, such that

$$\|f - p_k\|_{L_p(I_{kn}^*)} \leqslant C_2 \Lambda_{r,p}(f, |I_{kn}^*|, I_{kn}^*) \tag{6.10}$$

with $C_2 \neq C_2(f, n, k, I_{kn}^*)$.

Now define an approximating piecewise polynomial/spline by

$$L_n[f](x) := p_0(x)\, \theta_{0n}(x) + \sum_{k=1}^{n-1} (p_k - p_{k-1})(x)\, \theta_{kn}(x). \tag{6.11}$$

We first show that $L_n[f]$ is a good approximation to f:

LEMMA 6.1. *Let $\Psi_n : [-a_n, a_n] \to \mathbb{R}$ be such that uniformly in n, and $x \in [-a_n, a_n]$,*

$$\Psi_n(x) \sim \sqrt{1 - \frac{|x|}{a_{2n}}}. \tag{6.12}$$

Then for $0<p<\infty$,

$$\begin{aligned}
&\|(f-L_n[f])\,W\|^p_{L_p(\mathbb{R})}\\
&\quad\leqslant C_1\left\{\frac{n}{a_n}\int_0^{C_2(a_n/n)}\|W\Delta^r_{h\Psi_n(x)}(f,x,\mathbb{R})\|^p_{L_p[-a_n,a_n]}\,dh+\|fW\|^p_{L_p(|x|\geqslant a_n)}\right\}\\
&\quad\leqslant C_3\Big(\sup_{0<h\leqslant C_2(a_n/n)}\|W\Delta^r_{h\Psi_n(x)}(f,x,\mathbb{R})\|^p_{L_p[-a_n,a_n]}+\|fW\|^p_{L_p(|x|\geqslant a_n)}\Big).
\end{aligned} \tag{6.13}$$

Here $C_j\neq C_j(f,n)$, $j=1,2,3$. Moreover, the constants are independent of $\{\Psi_n\}$, depending only on the constants in $\sim$ in (6.12). For $p=\infty$, (6.13) holds if we remove the exponents p.

Proof. We first deal with $p<\infty$. Now

$$\|(f-L_n[f])\,W\|^p_{L_p(\mathbb{R})}=\sum_{j=0}^{n-1}\Delta_{jn}+\|fW\|^p_{L_p(|x|\geqslant a_n)}, \tag{6.14}$$

where

$$\Delta_{jn}:=\int_{I_{jn}}|f-L_n[f]|^p\,W^p. \tag{6.15}$$

Note that in $(\tau_{jn},\tau_{j+1,n})$, $L_n[f]=p_j$, so that

$$\begin{aligned}
\Delta_{jn}&=\int_{I_{jn}}|f-p_j|^p\,W^p\\
&\leqslant\|W\|^p_{L_\infty(I_{jn})}\,C_2^p\Delta^p_{r,p}(f,|I^*_{jn}|,I^*_{jn})\qquad\text{(by (6.10))}\\
&\leqslant\|W\|^p_{L_\infty(I^*_{jn})}\,\|W^{-1}\|^p_{L_\infty(I^*_{jn})}\frac{C_3}{|I^*_{jn}|}\int_0^{|I^*_{jn}|}\int_{I^*_{jn}}|W\Delta^r_s(f,x,I^*_{jn})|^p\,dx\,ds,
\end{aligned} \tag{6.16}$$

by (6.2), (6.3). Now from (3.11) of Lemma 3.2(a),

$$\|W\|^p_{L_\infty(I^*_{jn})}\,\|W^{-1}\|^p_{L_\infty(I^*_{jn})}\sim 1 \tag{6.17}$$

uniformly in j and n. Moreover, uniformly in j, n, and $x\in I^*_{jn}$,

$$|I^*_{jn}|\sim\frac{a_n}{n}\sqrt{1-\frac{|x|}{a_{2n}}}\sim\frac{a_n}{n}\,\Psi_n(x).$$

Then we can continue (6.16) as

$$\begin{aligned}\Delta_{jn} &\leqslant \frac{C_4}{|I^*_{jn}|} \int_{I^*_{jn}} \int_0^{|I^*_{jn}|} |W\Delta^r_s(f, x, I^*_{jn})|^p \, ds \, dx \\ &= \frac{C_4}{|I^*_{jn}|} \int_{I^*_{jn}} \Psi_n(x) \int_0^{|I^*_{jn}|/\Psi_n(x)} |W\Delta^r_{t\Psi_n(x)}(f, x, I^*_{jn})|^p \, dt \, dx \\ &\leqslant C_5 \frac{n}{a_n} \int_0^{C_6(a_n/n)} \int_{I^*_{jn}} |W\Delta^r_{t\Psi_n(x)}(f, x, I^*_{jn})|^p \, dx \, dt. \qquad (6.18)\end{aligned}$$

Adding over j gives

$$\sum_{j=0}^{n-1} \Delta_{jn} \leqslant C_5 \frac{n}{a_n} \int_0^{C_6(a_n/n)} \int_{I_n} |W\Delta^r_{t\Psi_n(x)}(f, x, \mathbb{R})|^p \, dx \, dt.$$

This and (6.14) give the result. Note that we have also effectively shown that

$$\begin{aligned}&\sum_{j=0}^{n-1} \Omega^p_{r,p}(f, |I^*_{jn}|, I^*_{jn}) \, W^p(\tau_{jn}) \\ &\qquad \leqslant C_5 \frac{n}{a_n} \int_0^{C_6(a_n/n)} \int_{I_n} |W\Delta^r_{t\Psi_n(x)}(f, x, \mathbb{R})|^p \, dx \, dt. \qquad (6.19)\end{aligned}$$

For $p = \infty$, the proof is similar, but easier: We see that

$$\begin{aligned}&\|(f - L_n[f]) \, W\|_{L_\infty(\mathbb{R})} \\ &\leqslant \max\Big\{ \max_{0 \leqslant j \leqslant n-1} \|(f - p_j) \, W\|_{L_\infty(I_{jn})}, \|fW\|_{L_\infty(|x| \geqslant a_n)} \Big\}.\end{aligned}$$

The rest of the proof is as before. ∎

Now we can define our polynomial approximation to f:

$$P_n[f] := p_0(x) \, R_{n, \tau_{on}}(x) + \sum_{k=1}^{n-1} (p_k - p_{k-1})(x) \, R_{n, \tau_{kn}}(x). \qquad (6.20)$$

Note that this has been formed from $L_n[f]$ of (6.11) by replacing the characteristic function $\theta_{kn}(x) = \chi_{[\tau_{kn}, a_n]}(x)$ by its polynomial approximation $R_{n, \tau_{kn}}(x)$ formed in the previous section.

LEMMA 6.2. *Let $\{\Psi_n\}_n$ be as in the previous lemma. Then for $0<p<\infty$,*

$$\|(L_n[f]-P_n[f])\,W\|_{L_p(\mathbb{R})}$$
$$\leqslant C\left\{\left(\frac{n}{a_n}\int_0^{C_1 a_n/n}\|W\Delta^r_{h\Psi_n(x)}(f,x,\mathbb{R})\|^p_{L_p[-a_n,a_n]}\,dh\right)^{1/p}+\|fW\|_{L_p(I^*_{0n})}\right\}. \tag{6.21}$$

For $p=\infty$, this remains valid if we replace the pth powers by appropriate sup norms.

Proof. We see that if we define $p_{-1}(x)\equiv 0$,

$$(L_n[f]-P_n[f])(x)=\sum_{k=0}^{n-1}(p_k-p_{k-1})(x)(\theta_{kn}(x)-R_{n,\tau_{kn}}(x)). \tag{6.22}$$

We shall make substantial use of the following inequality: Let S be a polynomial of degree at most r, and $[a,b]$ be a real interval. Then for all $x\in\mathbb{R}$,

$$|S(x)|\leqslant C(b-a)^{-1/p}\left(1+\frac{\min\{|x-a|,|x-b|\}}{b-a}\right)^r\|S\|_{L_p[a,b]}. \tag{6.23}$$

Here $C\neq C(a,b,x,S)$ but $C=C(p,r)$. This follows from standard Nikolskii inequalities and the Bernstein–Walsh inequality. See, for example, [24, p. 193]. Hence for $x\in\mathbb{R}$, and $1\leqslant k\leqslant n-1$,

$$|p_k-p_{k-1}|(x)\leqslant C\,|I_{kn}|^{-1/p}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^r\|p_k-p_{k-1}\|_{L_p(I_{kn})}.$$

This is still true for $k=0$ if we recall that $p_{-1}\equiv 0$. Now for $1\leqslant k\leqslant n-1$, (6.10) gives

$$\|p_k-p_{k-1}\|_{L_p(I_{kn})}\leqslant C_1\sum_{i=k-1}^{k}\Lambda_{r,p}(f,|I^*_{in}|,I^*_{in}),$$

where $C_1\neq C_1(f,k,n)$. This remains true for $k=0$ if we set

$$|I_{-1,n}|:=|I_{0n}|;\qquad |I^*_{-1,n}|:=|I^*_{0n}|;\qquad \tau_{-1,n}:=\tau_{0n};$$

and

$$\Lambda_{r,p}(f,|I^*_{-1,n}|,I^*_{-1,n}):=\|f\|_{L_p(I^*_{0n})}=:\Omega_{r,p}(f,|I^*_{-1,n}|,I^*_{-1,n}).$$

Since (see (3.6), (3.7), (6.6)) uniformly in k, n, and $x \in \mathbb{R}$,

$$1+\frac{|x-\tau_{kn}|}{|I_{kn}|} \sim 1+\frac{|x-\tau_{k-1,n}|}{|I_{k-1,n}|}$$

we obtain from (6.23) and Theorem 5.1, uniformly for $0 \leqslant k \leqslant n-1$ and $x \in \mathbb{R}$,

$$\begin{aligned} &|(p_k-p_{k-1})(x)(\theta_{kn}(x)-R_{n,\tau_{kn}}(x))| \frac{W(x)}{W(\tau_{kn})} \\ &\quad \leqslant C_2 \sum_{i=k-1}^{k} |I_{in}|^{-1/p} \left(1+\frac{|x-\tau_{in}|}{|I_{in}|}\right)^{r-l} \Omega_{r,p}(f, |I_{in}^*|, I_{in}^*). \end{aligned} \tag{6.24}$$

We consider three different ranges of p:

(I) $0<p<1$. Here from (6.22) and then (6.24),

$$\begin{aligned} \int_{\mathbb{R}} (|L_n[f]-P_n[f]|\, W)^p &\leqslant \sum_{k=0}^{n-1} \int_{\mathbb{R}} (|p_k-p_{k-1}|\, |\theta_{kn}-R_{n,\tau_{kn}}|\, W)^p \\ &\leqslant \sum_{k=-1}^{n-1} |I_{kn}|^{-1}\, \Omega_{r,p}^p(f, |I_{kn}^*|, I_{kn}^*)\, W^p(\tau_{kn}) \\ &\quad \times \int_{\mathbb{R}} \left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{(r-l)p} dx. \end{aligned} \tag{6.25}$$

Here if $(r-l)p<-1$,

$$|I_{kn}|^{-1} \int_{\mathbb{R}} \left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{(r-l)p} dx = \int_{\mathbb{R}} (1+|u|)^{(r-l)p}\, du =: C_3 < \infty.$$

So

$$\int_{\mathbb{R}} (|L_n[f]-P_n[f]|\, W)^p \leqslant C_4 \sum_{k=-1}^{n-1} \Omega_{r,p}^p(f, |I_{kn}^*|, I_{kn}^*)\, W^p(\tau_{kn}).$$

This is the same as our sum in (6.19) except for the term for $k=-1$. So the estimate (6.19) gives the estimate (6.21), keeping in mind our choice of $\Omega_{r,p}(f, |I_{-1,n}^*|, I_{-1,n}^*)$.

(II) $1 \leqslant p < \infty$. From (6.22) and (6.24) and then Hölder's inequality,

$$\{|L_n[f]-P_n[f]|\,(x)\,W(x)\}^p$$
$$\leqslant C\left\{\sum_{k=-1}^{n-1} |I_{kn}|^{-1/p}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{r-l} \Omega_{r,p}(f,|I_{kn}^*|,I_{kn}^*)\,W(\tau_{kn})\right\}^p$$
$$\leqslant C\sum_{k=-1}^{n-1} |I_{kn}|^{-1}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{(r-l)p/2}$$
$$\times\,\Omega_{r,p}^p(f,|I_{kn}^*|,I_{kn}^*)\,W^p(\tau_{kn})\cdot S_n(x)^{p/q}, \tag{6.26}$$

where $q := p/(p-1)$ and

$$S_n(x) := \sum_{k=0}^{n-1}\left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{(r-l)q/2}.$$

We shall show that if $(r-l)q/2 < -1$, then

$$\sup_{n\geqslant 1}\,\sup_{x\in\mathbb{R}} S_n(x) \leqslant C_1 < \infty. \tag{6.27}$$

Note that $S_n(x)$ is a decreasing function of x for $x \geqslant a_n = \tau_{nn}$, so it suffices to consider $x\in[0, a_n]$. Recall that

$$|I_{kn}| \sim |I_{k+1,n}| \sim \frac{a_n}{n}\sqrt{1-\frac{|\tau_{kn}|}{a_{2n}}}.$$

It is then not difficult to see that

$$S_n(x) \leqslant C_2\frac{n}{a_n}\int_{-a_n}^{a_n}\left(1+\frac{n}{a_n}\frac{|x-u|}{\sqrt{1-|u|/a_{2n}}}\right)^{(r-l)q/2}\frac{du}{\sqrt{1-|u|/a_{2n}}}$$
$$\leqslant C_3 n\int_{-1}^{1}\left(1+\frac{n\,|\bar{x}-s|}{\sqrt{1-s}}\right)^{(r-l)q/2}\frac{ds}{\sqrt{1-s}},$$

where $\bar{x} := x/a_{2n}$, so that

$$1-\bar{x} \geqslant 1-a_n/a_{2n} \geqslant C_4 T(a_n)^{-1} \geqslant C_5 n^{-2}.$$

We make the substitution $(1-s)=(1-\bar{x})w$ to obtain

$$S_n(x) \leqslant C_3 n\sqrt{1-\bar{x}} \int_0^{2/(1-\bar{x})} \left(1+n\sqrt{1-\bar{x}}\,\frac{|w-1|}{\sqrt{w}}\right)^{(r-l)q/2} \frac{dw}{\sqrt{w}}$$

$$\leqslant C_4 n\sqrt{1-\bar{x}} \left\{ \int_0^{1/2} \left[1+\frac{n\sqrt{1-\bar{x}}}{\sqrt{w}}\right]^{(r-l)q/2} \frac{dw}{\sqrt{w}} \right.$$

$$+\int_{1/2}^{3/2} [1+n\sqrt{1-\bar{x}}\,|w-1|]^{(r-l)q/2}\,dw$$

$$\left. +\int_{3/2}^{2/(1-\bar{x})} [1+n\sqrt{(1-\bar{x})w}]^{(r-l)q/2} \frac{dw}{\sqrt{w}} \right\}.$$

(We can omit the third integral if $2/(1-\bar{x}) \leqslant 3/2$.) We now make the substitutions $w=n^2(1-\bar{x})v$ in the first integral, $v=n\sqrt{1-\bar{x}}\,(w-1)$ in the second integral, and $v=n^2(1-\bar{x})w$ in the third integral. It is then not difficult to see that the resulting terms are bounded independent of n and x if l is large enough. (The least obvious is the first integral: there we need to ensure that $(r-l)\,q/4-1/2 \geqslant 0$, so that the integrand is bounded after the substitution.) So we have (6.27). Then integrating (6.26) and using (6.19) gives our result.

(III) $p=\infty$. Now

$$|L_n[f]-P_n[f]|(x)$$

$$\leqslant C \sum_{k=0}^{n-1} |p_k-p_{k-1}|\,(x)\,|\theta_{kn}-R_{n,\tau_{kn}}|\,(x)\,W(x)$$

$$\leqslant C \max_{-1 \leqslant k \leqslant n-1} \Omega_{r,p}(f, |I_{kn}^*|, I_{kn}^*)\,W(\tau_{kn}) \cdot \sum_{k=0}^{n-1} \left(1+\frac{|x-\tau_{kn}|}{|I_{kn}|}\right)^{(r-l)}.$$

As before, the sum is bounded if l is large enough. Then we can continue this as

$$\leqslant C_1 \{ \sup_{0 \leqslant k \leqslant n-1} \sup_{0<h \leqslant |I_{kn}^*|} \|\Delta_h^r(f, x, I_{kn}^*)\,W\|_{L_\infty(I_{kn}^*)} + \|fW\|_{L_\infty(I_{0n}^*)} \}$$

$$\leqslant C_2 \{ \sup_{0 \leqslant k \leqslant n-1} \sup_{0<h \leqslant Ca_n/n} \|\Delta_{h\Psi_n(x)}^r(f, x, I_{kn}^*)\,W\|_{L_\infty(I_{kn}^*)} + \|fW\|_{L_\infty(I_{0n}^*)} \}$$

$$\leqslant C_3 \{ \sup_{0<h \leqslant Ca_n/n} \|\Delta_{h\Psi_n(x)}^r(f, x, \mathbb{R})\,W\|_{L_\infty(-a_n, a_n)} + \|fW\|_{L_\infty(I_{0n}^*)} \}. \quad \blacksquare$$

We can now turn to the

Proof of Theorem 1.2. Now recall that $R_{n,\tau}$ has degree at most $2lJn$, where J is as in the proof of Theorem 5.1. So $P_n[f]$ has degree at most $2lJn+r$. So, if $M:=3lJ$, we have for large n,

$$\begin{aligned} E_{Mn}[f]_{W,p} &\leqslant \|(f-P_n[f])\,W\|_{L_p(\mathbb{R})} \\ &\leqslant C\{\|(f-L_n[f])\,W\|_{L_p(\mathbb{R})}+\|(L_n[f]-P_n[f])\,W\|_{L_p(\mathbb{R})}\} \\ &\leqslant C_1\left\{\left(\frac{n}{a_n}\int_0^{C_2a_nn/n}\|\Delta^r_{h\Psi_n(x)}(f,x,\mathbb{R})\,W\|^p_{L_p(-a_n,a_n)}\,dh\right)^{1/p}\right. \\ &\quad\left.+\|fW\|_{L_p(|x|\geqslant a_n(1-C_2[nT(a_n)^{1/2}]^{-1}))}\right\}. \end{aligned} \tag{6.28}$$

Here we have used Lemmas 6.1 and 6.2, and also (6.6), which implies that

$$|I^*_{0n}|\sim\frac{a_n}{n}\sqrt{1-\frac{a_n}{a_{2n}}}\sim\frac{a_n}{n}\,T(a_n)^{-1/2}.$$

Furthermore, at this stage, the functions $\{\Psi_n\}$ are any functions satisfying (6.12): they will be explicitly chosen later. Next for

$$Mn\leqslant j\leqslant M(n+1) \tag{6.29}$$

we write

$$n=\kappa j,$$

where $\kappa=\kappa(j,n)$. Note that

$$\kappa=\frac{n}{j}\to\frac{1}{M},\qquad j\to\infty. \tag{6.30}$$

We set

$$t:=t(j):=\frac{Ma_j}{3j}.$$

Note that then

$$\frac{t}{a_n/n}=\frac{1}{3}\,\frac{Mn}{j}\,\frac{a_j}{a_n}=\frac{1}{3}\,(1+o(1)),\qquad n\to\infty. \tag{6.31}$$

Let $\beta > 3$. We claim that for large enough n,

$$a_n(1 - C_2[nT(a_n)^{1/2}]^{-1}) \geqslant \sigma(\beta t). \tag{6.32}$$

To see this, note from (2.8) that

$$[nT(a_n)^{1/2}]^{-1} = o(T(a_n)^{-1})$$

so that by (2.7), if $1 > \alpha > 3/\beta$,

$$\begin{aligned} a_n(1 - C_2[nT(a_n)^{1/2}]^{-1}) &\geqslant a_n\left(1 - o\left(\frac{1}{T(a_n)}\right)\right) \geqslant a_{\alpha n} \\ &\geqslant \sigma\left(\frac{a_{\alpha n}}{\alpha n}\right) = \sigma\left(\frac{3t}{\alpha}[1 + o(1)]\right) \geqslant \sigma(\beta t), \end{aligned}$$

for large enough j, by first (3.2) and then (6.31). Next, we claim that if $0 < \gamma < 3$, then for n large enough,

$$a_n \leqslant \sigma(\gamma t). \tag{6.33}$$

To see this, note that by (6.31) if $1 < \delta < 3/\gamma$

$$\sigma(\gamma t) = \sigma\left(\frac{\gamma a_n}{3n}[1 + o(1)]\right) \geqslant \sigma\left(\frac{a_{\delta n}}{\delta n}\right) = a_{\delta n(1+o(1))} \geqslant a_n.$$

Here we also used the fact that σ is decreasing, and also (3.2), (3.3) with n large enough. Since also $a_n/n \leqslant 4t$ for large enough n, we can recast (6.28) as

$$\begin{aligned} E_j[f]_{W,p} &\leqslant E_{Mn}[f]_{W,p} \\ &\leqslant C_1\left\{\left(\frac{1}{2t}\int_0^{4Ct} \|\Delta^r_{h\Psi_n(x)}(f, x, \mathbb{R})\, W\|^p_{L_p(-\sigma(2t),\, \sigma(2t))}\, dh\right)^{1p} \right. \\ &\quad \left. + \|fW\|_{L_p(|x| \geqslant \sigma(4t))}\right\}. \end{aligned} \tag{6.34}$$

We now turn to our choice of $\{\Psi_n\}$: we must ensure that (6.12) holds with constants independent of x, j, and n, that is,

$$\Psi_n(x) \sim \sqrt{1 - \frac{|x|}{a_{2n}}}, \qquad |x| \leqslant a_n.$$

But for this range of x,

$$\sqrt{1-\frac{|x|}{a_{2n}}} \sim \sqrt{1-\frac{|x|}{a_{2n}}} + T(a_{2n})^{-1/2} \sim \Phi_{(a_{2n}/2n)}(x) \sim \Phi_t(x)$$

by Lemma 3.1(d), (e). We choose $h_1 := h/(4C)$ and $\Psi_n := \Phi_t/(4C)$ so that $h\Psi_n = h_1 \Phi_t$, a choice satisfying (6.12). Then we rewrite (6.34) as

$$E_j[f]_{W,p} \leqslant C_1 \left\{ \left(\frac{4C}{2t} \int_0^t \|\Delta^r_{h_1 \Phi_t(x)}(f, x, \mathbb{R}) W\|^p_{L_p(-\sigma(2t), \sigma(2t))}\, dh_1 \right)^{1/p} + \|fW\|_{L_p(|x| \geqslant \sigma(4t))} \right\}.$$

Replacing f by $f - P_0$ with a suitable choice of $P_0 \in \mathscr{P}_{r-1}$, we have for large enough j,

$$\begin{aligned} E_j[f]_{W,p} &= E_j[f-P_0]_{W,p} \\ &\leqslant C_3 \left\{ \left(\frac{1}{t} \int_0^t \|\Delta^r_{h_1 \Phi_t(x)}(f, x, \mathbb{R}) W\|^p_{L_p(-\sigma(2t), \sigma(2t))}\, dh_1 \right)^{1/p} + \|(f-P_0) W\|_{L_p(|x| \geqslant \sigma(4t))} \right\} \\ &\leqslant 2C_3 \left\{ \left(\frac{1}{t} \int_0^t \|\Delta^r_{h_1 \Phi_t(x)}(f, x, \mathbb{R}) W\|^p_{L_p(-\sigma(2t), \sigma(2t))}\, dh_1 \right)^{1/p} + \inf_{P \in \mathscr{P}_{r-1}} \|(f-P) W\|_{L_p(|x| \geqslant \sigma(4t))} \right\} \\ &= C_3 \bar{\omega}_{r,p}(f, W, t) = C_3 \bar{\omega}_{r,p}\left(f, W, \frac{Ma_j}{3j}\right). \quad \blacksquare \end{aligned}$$

For use in [3], we record the following form of Theorem 1.2:

THEOREM 6.3. *For $n \geqslant 1$, let $\lambda(n) \in [\frac{4}{5}, 1]$. Then*

$$E_n[f]_{W,p} \leqslant C_1 \bar{\omega}_{r,p}\left(f, W, C_2 \lambda(n) \frac{a_n}{n}\right), \tag{6.35}$$

where C_1, C_2 do not depend on n or f or $\{\lambda(n)\}$. Moreover,

$$E_n[f]_{W,p} \leqslant C_1 \inf_{\rho \in [4/5, 1]} \bar{\omega}_{r,p}\left(f, W, C_2 \rho \frac{a_n}{n}\right). \tag{6.36}$$

Proof. Obviously (6.36) implies (6.35). The only difference to the above proof is that for $\rho \in [\frac{4}{5}, 1]$, we choose

$$t_1 := \rho t := \rho \frac{Ma_j}{3j}$$

to replace t above. Then from (6.31),

$$\frac{t_1}{a_n/n} = \frac{\rho}{3}(1 + o(1))$$

and here $\rho/3 \in [\frac{4}{15}, \frac{1}{3}]$. Then as $4\rho > 3$, (6.32) above shows that

$$a_n(1 - C_2[nT(a_n)^{1/2}]^{-1}) \geqslant \sigma(4\rho t) = \sigma(4t_1)$$

and as $\rho \leqslant 1$, (6.33) above shows that

$$a_n \leqslant \sigma(2\rho t) = \sigma(2t_1).$$

Moreover, for large enough n, $a_n/n \leqslant 3t(1 + o(1)) \leqslant 4t_1$. Choosing $h_1 := h/(4C)$ and $\Psi_n(x) := \Phi_{t_1}(x)/(4C)$ we note that (6.12) holds uniformly in ρ. We proceed as before to obtain

$$E_j[f]_{W,p} \leqslant C_1 \bar{\omega}_{r,p}\left(f, W, C_2 \frac{\rho a_j}{j}\right)$$

with constants independent of ρ, f, j. ∎

7. THE PROOF OF THEOREM 1.3

We begin with a technical lemma, which refines part of Lemma 3.1:

LEMMA 7.1. (a) *For* $0 < s < t \leqslant C$,

$$T(\sigma(t))\left(1 - \frac{\sigma(t)}{\sigma(s)}\right) \leqslant C_1 \log\left(2 + \frac{t}{s}\right). \tag{7.1}$$

(b) *For* $0 < s < t \leqslant C$,

$$\sup_{x \in \mathbb{R}} \frac{\Phi_s(x)}{\Phi_t(x)} \leqslant C_2 \sqrt{\log\left(2 + \frac{t}{s}\right)}. \tag{7.2}$$

Hence, given $\gamma > 0$,

$$\sup_{x \in \mathbb{R}} \left(\frac{s}{t}\right)^{\gamma} \frac{\Phi_s(x)}{\Phi_t(x)} \leqslant C_3. \tag{7.3}$$

Proof. (a) We write $s = a_u/u$ and $t = a_v/v$. Note (with the notation of Lemma 3.1) that

$$a_{\beta(u)} = \sigma(s) \geqslant \sigma(t) = a_{\beta(v)},$$

so $\beta(u) \geqslant \beta(v)$. Using the inequality

$$1 - u \leqslant \log \frac{1}{u}, \qquad u \in (0, 1]$$

we obtain

$$\begin{aligned} 1 - \frac{\sigma(t)}{\sigma(s)} &\leqslant \log \frac{\sigma(s)}{\sigma(t)} = \log \frac{a_{\beta(u)}}{a_{\beta(v)}} \\ &\leqslant C_1 \frac{\log C(\beta(u)/\beta(v))}{T(a_{\beta(v)})} = C_1 \frac{\log C(\beta(u)/\beta(v))}{T(\sigma(t))} \end{aligned} \tag{7.4}$$

by (2.10). Next, $\beta(u) = u(1 + o(1))$, and similarly for $\beta(v)$, so it suffices to show that

$$\log \frac{u}{v} \leqslant C_2 \log \left(2 + \frac{t}{s}\right). \tag{7.5}$$

But from (2.1) for $s < t$ and small t, and then from (2.5),

$$\begin{aligned} \frac{u}{v} \bigg/ \frac{t}{s} &= \frac{a_u}{a_v} \leqslant \left(\frac{Q(a_u)}{Q(a_v)}\right)^{1/2} \\ &\leqslant C_1 \left(\frac{uT(a_u)^{-1/2}}{vT(a_v)^{-1/2}}\right)^{1/2} \leqslant C_2 \left(\frac{uT(a_{\beta(u)})^{-1/2}}{vT(a_{\beta(v)})^{-1/2}}\right)^{1/2} \leqslant C_3 \left(\frac{u}{v}\right)^{1/2} \end{aligned}$$

as $\beta(u) \geqslant \beta(v)$. So

$$\left(\frac{u}{v}\right)^{1/2} \leqslant C_4 \frac{t}{s} \tag{7.6}$$

and we have (7.5).

(b) Now if $x \geqslant 0$,

$$\left|1-\frac{x}{\sigma(s)}\right| \leqslant \left|1-\frac{x}{\sigma(t)}\right| + \frac{x}{\sigma(t)}\left|1-\frac{\sigma(t)}{\sigma(s)}\right|$$

$$\leqslant \left|1-\frac{x}{\sigma(t)}\right| + \left(\left|1-\frac{x}{\sigma(t)}\right|+1\right)\left|1-\frac{\sigma(t)}{\sigma(s)}\right|.$$

Using (a) of this lemma, we obtain

$$\left|1-\frac{x}{\sigma(s)}\right|^{1/2} \leqslant C_{12}\Phi_t(x)\sqrt{\log\left(2+\frac{t}{s}\right)}.$$

Since $\sigma(s) \geqslant \sigma(t)$, also

$$T(\sigma(s))^{-1/2} \leqslant C_{13}\, T(\sigma(t))^{-1/2}.$$

So (7.2) follows. ∎

We turn to the proof of Theorem 1.3. We provide full proofs only where the details are significantly different, and otherwise refer back. We begin with an analogue of Lemma 6.1 for $L_n[f]$ of (6.11).

Lemma 7.2.

$$\|(f-L_n[f])\,W\|_{L_p(\mathbb{R})} \leqslant C_1\Big[\sup_{\substack{0<h\leqslant a_{3n}/(3n)\\ 0<\tau\leqslant L}} \|W\Delta^r_{\tau h\Phi_h(x)}(f, x, \mathbb{R})\|_{L_p[-a_n, a_n]}$$

$$+\|fW\|_{L_p(|x|\geqslant a_n)}\Big]. \tag{7.7}$$

Here L is independent of f, n.

Proof. We do this for $p<\infty$. Recall that the crux of Lemma 6.1 is estimation of

$$\Delta_{jn} := \int_{I_{jn}} |f-p_j|^p\, W^p \leqslant C_1\Omega_{r,p}(f, |I^*_{jn}|, I^*_{jn})^p\, W^p(\tau_{jn})$$

$$\leqslant \frac{C_2}{|I^*_{jn}|}\int_{I^*_{jn}}\int_0^{|I^*_{jn}|} |W\Delta^r_s(f, x, I^*_{jn})|^p\, ds\, dx. \tag{7.8}$$

We now choose $L>0$ such that for $0<h\leqslant 1$,

$$\sup_{x\in\mathbb{R}} \frac{\frac{h}{L}\Phi_{\frac{h}{L}}(x)}{h\Phi_h(x)} \leqslant \frac{1}{2}. \tag{7.9}$$

This is possible by (7.2). Now we choose

$$\delta_{n,k}(x) := L^{1-k}\frac{a_{3n}}{3n}\,\Phi_{L^{1-k}(a_{3n}/3n)}(x), \qquad k \geqslant 1.$$

Note that by (7.9),

$$\sup_{x\in\mathbb{R}} \frac{\delta_{n,k+1}(x)}{\delta_{n,k}(x)} \leqslant \frac{1}{2}. \tag{7.10}$$

In view of (6.6), (3.6), and (3.7), we may assume that L is so large that uniformly in n, j, $x \in I_{jn}^*$,

$$|I_{jn}^*| \leqslant L\frac{a_{3n}}{3n}\,\Phi_{a_{3n}/3n}(x) = L\delta_{n,1}(x); \qquad |I_{jn}^*| \sim \delta_{n,1}(x).$$

Then from (7.8),

$$\begin{aligned}
\Delta_{jn} &\leqslant C_4 \int_{I_{jn}^*}\int_0^{L\delta_{n,1}(x)} \frac{1}{\delta_{n,1}(x)}\,|W\Delta_s^r(f, x, I_{jn}^*)|^p\,ds\,dx\\
&= C_4 \int_{I_{jn}^*}\sum_{k=1}^{\infty}\int_{L\delta_{n,k+1}(x)}^{L\delta_{n,k}(x)} \frac{1}{\delta_{n,1}(x)}\,|W\Delta_s^r(f, x, I_{jn}^*)|^p\,ds\,dx\\
&= C_4 \int_{I_{jn}^*}\sum_{k=1}^{\infty}\int_{L\delta_{n,k+1}(x)/\delta_{n,k}(x)}^{L} \frac{\delta_{n,k}(x)}{\delta_{n,1}(x)}\,|W\Delta_{\tau\delta_{n,k}(x)}^r(f, x, I_{jn}^*)|^p\,d\tau\,dx\\
&\leqslant C_4 \int_{I_{jn}^*}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}\int_0^{L} |W\Delta_{\tau\delta_{n,k}(x)}^r(f, x, I_{jn}^*)|^p\,d\tau\,dx.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{j=0}^{n-1}\Delta_{jn} &\leqslant C_4 \int_{-a_n}^{a_n}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}\int_0^{L} |W\Delta_{\tau\delta_{n,k}(x)}^r(f, x, \mathbb{R})|^p\,d\tau\,dx\\
&\leqslant 2C_4 \sup_{\substack{0<h\leqslant a_{3n}/(3n)\\ 0<\tau\leqslant L}} \int_{-a_n}^{a_n} |W\Delta_{\tau h\Phi_h(x)}^r(f, x, \mathbb{R})|^p\,dx.
\end{aligned}$$

The rest of the proof is as before. ∎

The analogue of Lemma 6.2 is

LEMMA 7.3.

$$\|(L_n[f]-P_n[f])\,W\|_{L_p(\mathbb{R})} \leqslant C_1 \Big\{ \sup_{\substack{0<h\leqslant a_{3n}/(3n)\\ 0<\tau\leqslant L}} \|W\Delta^r_{\tau h\Phi_h(x)}(f, x, \mathbb{R})\|_{L_p[-a_n, a_n]} + \|fW\|_{L_p(I^*_{0n})}\Big\}.$$

Proof. This is exactly the same as the proof of Lemma 6.2, except that we substitute for (6.19) the estimate of Lemma 7.2. ∎

Proof of Theorem 1.3. This follows from Lemmas 7.2 and 7.3 exactly as Theorem 1.2 followed from Lemmas 6.1 and 6.2. ∎

Finally, we briefly show that under some additional conditions on Q, we can use the simpler modulus

$$\omega^{\#}_{r,p}(f, W, t) = \sup_{0<h\leqslant t} \|W\Delta^r_{Lh\Phi_h(x)}(f, x, \mathbb{R})\|_{L_p(|x|\leqslant\sigma(2h))} + \inf_{P\in\mathcal{P}_{r-1}} \|(f-P)\,W\|_{L_p(|x|\geqslant\sigma(4t))}. \tag{7.11}$$

We shall assume in addition to $W\in\mathcal{E}$ that Q'' exists and is non-negative in $(0, \infty)$, and

$$\frac{Q''(x)}{Q'(x)} \sim \frac{Q'(x)}{Q(x)}, \qquad x\in(0, \infty). \tag{7.12}$$

Moreover, we assume that

$$|T'(x)| \leqslant C_1 \frac{T^2(x)}{x}, \qquad x\geqslant C_1. \tag{7.13}$$

Using (7.12) and the methods of proof of Lemma 2.2 in [13, p. 209], we obtain

$$\frac{a'_u}{a_u} \sim \frac{1}{uT(a_u)}, \qquad u\geqslant C_2 \tag{7.14}$$

and hence

$$\frac{d}{du}\left(\frac{a_u}{u}\right) \sim -\frac{a_u}{u^2}, \qquad u\geqslant C_2. \tag{7.15}$$

Since $u \to a_u/u$ is then strictly decreasing for large u, we obtain the identity

$$\sigma\left(\frac{a_u}{u}\right) = a_u, \qquad u \geqslant C_3. \tag{7.16}$$

Differentiating this and using (7.14), (7.15) lead to

$$\frac{\sigma'(t)}{\sigma(t)} \sim -\frac{1}{tT(\sigma(t))}, \qquad 0 < t \leqslant C_4 \tag{7.17}$$

and then using (7.13), we obtain

$$\left| t \frac{d}{dt} T(\sigma(t)) \right| \leqslant C_5 T(\sigma(t)), \qquad 0 < t \leqslant C_4. \tag{7.18}$$

These last two bounds easily give

$$\left| \frac{d}{dt} [t\Phi_t(x)] \right| \leqslant C_5 \Phi_t(x) \tag{7.19}$$

for

$$0 < t \leqslant C_5; \qquad \left| 1 - \frac{|x|}{\sigma(t)} \right| \geqslant \frac{\varepsilon}{T(\sigma(t))}. \tag{7.20}$$

Here ε is any fixed positive number. We now estimate Δ_{jn} a little differently from the way we proceeded after (7.8). Let us make the substitution $s = Lt\Phi_t(x)$ in the right-hand side of (7.8) and keep our choice of L, $\delta_{n,1}(x)$ to deduce that

$$\begin{aligned}\Delta_{jn} &\leqslant C_6 \int_{I_{jn}^*} \int_0^{a_{3n}/(3n)} \frac{1}{\delta_{n,1}(x)} |W\Delta^r_{Lt\Phi_t(x)}(f, x, I_{jn}^*)|^p \left| \frac{d}{dt} [t\Phi_t(x)] \right| dt\, dx \\ &\leqslant \frac{C_7 3n}{a_{3n}} \int_{I_{jn}^*} \int_0^{a_{3n}/(3n)} \sqrt{\log\left(2 + \frac{a_{3n}}{3nt}\right)}\, |W\Delta^r_{Lt\Phi_t(x)}(f, x, I_{jn}^*)|^p \, dt\, dx\end{aligned}$$

by (7.19) and (7.2). In applying (7.19) we must ensure that the range conditions in (7.20) must hold for $x \in I_{jn}^*$ and $t \leqslant a_{3n}/(3n)$. In fact if $|x| \leqslant a_n$, then

$$\begin{aligned}1 - \frac{|x|}{\sigma(t)} &\geqslant 1 - \frac{a_n}{\sigma(a_{3n}/(3n))} \geqslant 1 - \frac{a_n}{a_{3n(1+o(1))}} \\ &\geqslant C_8 T(a_n)^{-1} \geqslant C_9 T(\sigma(t))^{-1}\end{aligned}$$

by (3.2), (3.3), then (2.7) and then (2.6). Thus

$$\sum_{j=0}^{n-1} \Delta_{jn} \leqslant \frac{C_8 3n}{a_{3n}} \int_{-a_n}^{a_n} \int_0^{a_{3n}/(3n)} \sqrt{\log\left(2+\frac{a_{3n}}{3nt}\right)}\, |W\Delta^r_{Lt\Phi_t(x)}(f, x, \mathbb{R})|^p \, dt\, dx$$

$$\leqslant C_8 \sup_{0<t\leqslant a_{3n}/(3n)} \int_{-a_n}^{a_n} |W\Delta^r_{Lt\Phi_t(x)}(f, x, \mathbb{R})|^p \, dx \int_0^1 \sqrt{\log\left(2+\frac{1}{s}\right)}\, ds.$$

So under the additional conditions on Q we obtain

$$E_n[f]_{W,p} \leqslant C_9 \omega^{\#}_{r,p}\left(f, W, C_{10}\frac{a_n}{n}\right). \tag{7.21}$$

We note that these additional conditions (7.12) and (7.13) are certainly satisfied for $W_{k,\alpha}$ of (1.6).

REFERENCES

1. Yu. A. Brudnyi, Approximation of functions by algebraic polynomials, *Izv. Akad. Nauk SSSR Ser. Math.* **32**, No. 4 (1968), 780–787.
2. J. Clunie and T. Kövari, On integral functions having prescribed asymptotic growth, II, *Canad. J. Math.* **20** (1968), 7–20.
3. S. B. Damelin, Converse and smoothness theorems for Erdős weights in $L_p(0<p\leqslant\infty)$, *J. Approx. Theory* **93** (1998), 349–398.
4. R. A. DeVore and V. A. Popov, Interpolation of Besov spaces, *Trans. Amer. Math. Soc.* **305** (1988), 397–414.
5. R. A. DeVore, D. Leviatan, and X. M. Yu, Polynomial approximation in $L_p(0<p<1)$, *Constr. Approx.* **8** (1992), 187–201.
6. Z. Ditzian and D. S. Lubinsky, Jackson and smoothness theorems for Freud weights in $L_p, 0<p\leqslant\infty$, *Constr. Approx.* **13** (1997), 99–152.
7. Z. Ditzian and V. Totik, Moduli of smoothness, *in* "Springer Series in Computational Mathematics," Vol. 9, Springer-Verlag, Berlin, 1987.
8. G. Freud, "Orthogonal Polynomials," Pergamon/Akademiai Kiado, Budapest, 1971.
9. M. von Golitschek, G. G. Lorentz, and Y. Makovoz, Asymptotics of weighted polynomials, *in* "Progress in Approximation Theory" (A. A. Gonchar and E. B. Saff, Eds.), Springer Series in Computational Mathematics, Vol. 19, pp. 431–451, Springer-Verlag, New York, 1992.
10. S. Jansche and R. L. Stens, Best weighted polynomial approximation on the real line: A functional analytic approach, *J. Comput. Appl. Math.* **40** (1992), 199–213.
11. D. Leviatan and X. M. Yu, Shape preserving approximation by polynomials in L_p, manuscript.
12. A. L. Levin and D. S. Lubinsky, Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights, *Constr. Approx.* **8** (1992), 463–535.
13. A. L. Levin, D. S. Lubinsky, and T. Z. Mthembu, Christoffel functions and orthogonal polynomials for Erdős weights on $(-\infty, \infty)$, *Rend. Mat. Appl.* **14** (1994), 199–289.
14. D. S. Lubinsky, Strong asymptotics for extremal errors and polynomials associated with Erdős type weights, *in* "Pitman Research Notes in Mathematics," Vol. 202, Longman, Harlow, Essex, 1989.

15. D. S. Lubinsky, An update on orthogonal polynomials and weighted approximation on the real line, *Acta Appl. Math.* **33** (1993), 121–164.
16. D. S. Lubinsky, Ideas of weighted polynomial approximation on $(-\infty, \infty)$, *Approximation and Interpolation*, World Scientific, (1995), pp. 371–396.
17. D. S. Lubinsky and T. Z. Mthembu, Orthogonal expansions and the error of weighted polynomial approximation for Erdős weights, *Numer. Funct. Anal. and Optimiz.* **13** (1992), 327–347.
18. H. N. Mhaskar and E. B. Saff, Extremal problems for polynomials with exponential weights, *Trans. Amer. Math. Soc.* **285** (1984), 203–234.
19. H. N. Mhaskar and E. B. Saff, Where does the sup-norm of a weighted polynomial live? *Constr. Approx.* **1** (1985), 71–91.
20. H. N. Mhaskar and E. B. Saff, Where does the L_p norm of a weighted polynomial live? *Trans. Amer. Math. Soc.* **303** (1987), 109–124.
21. P. Nevai, G. Freud, Orthogonal polynomials and Christoffel functions: A case study, *J. Approx. Theory* **48** (1986), 3–167.
22. P. Nevai (Ed.), "Orthogonal Polynomials, Theory and Practice," NATO ASI Series, Vol. 294, Kluwer Academic, Dordrecht, 1990.
23. P. Oswald, Ungleichungen vom Jackson-Typ für die algebraische beste Approximation in L_p, *J. Approx. Theory* **23**, No. 2 (1978), 113–136.
24. P. P. Petrushev and V. Popov, "Rational Approximation of Real Functions," Cambridge Univ. Press, Cambridge, 1987.
25. E. B. Saff and V. Totik, "Logarithmic Potentials with External Fields," Springer-Verlag, New York/Berlin, 1997.
26. V. Totik, Weighted approximation with varying weight, *in* "Lecture Notes in Mathematics," Vol. 1569, Springer-Verlag, Berlin, 1994.